Triangles with three rational medians

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Abstract

We present a characterization of all rational sided triangles with three rational medians. It turns out that they each correspond to a point on a one-parameter family of elliptic curves. It is possible to show that the rank of this family is at least two and in fact some reasonably high rank curves appear among them.

1 Introduction

During the 1770's Euler was working on (among many other things) triangles with the property that the distance from a vertex to the centre of gravity was rational. On 17 December 1778, he presented [EUL1778 p.111-113] a parametrization of such triangles in which all three such lengths were rational, namely,

$$\begin{aligned} a &= 2q(-9q^4 + 10qqrr + 3r^4) \\ b &= r(9q^4 - 6qqrr + r^4) - q(9q^4 + 26qqrr + r^4) \\ c &= r(9q^4 - 6qqrr + r^4) + q(9q^4 + 26qqrr + r^4). \end{aligned}$$

Since the distance from a vertex to the centre of gravity is two-thirds the length of the corresponding median, this immediately provides a parametrization of triangles with three rational medians. Unfortunately, this turned out to be incomplete since the triangle, $(466, 510, 884)$, with three integer medians, can not be represented in this way. The main result is the following ...

Theorem 1 *Every rational-sided triangle with three rational medians corresponds to a point on the one parameter elliptic curve*

$$E[\phi] : y^2 = x(x^2 + (\phi^2 + 1)^2 x - (8\phi(\phi^2 - 1)/3)^2)$$

for some $\phi \in \mathbb{Q}$*. Furthermore, the torsion subgroup of* $E[\phi](\mathbb{Q})$ *is precisely* $\mathbb{Z}/2\mathbb{Z}$ *while rank*$[E[\phi](\mathbb{Q})] \geq 2$.

A useful alternative view is that this is an elliptic surface defined over the field $\mathbb{Q}$, however we will continue to think of it as a curve over $\mathbb{Q}(\phi)$ for the rest of this paper.

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2 Deriving the elliptic curve

It is well known that the medians, k, l, m, and sides, a, b, c, of a triangle satisfy the following relationships

$$\begin{aligned} k^2 &= 2b^2 + 2c^2 - a^2 \\ l^2 &= 2c^2 + 2a^2 - b^2 \\ m^2 &= 2a^2 + 2b^2 - c^2. \end{aligned} \tag{1}$$

Factorizing the first of these equations over the field $\mathbb{Q}(\sqrt{2})$ gives us

$$(k - b\sqrt{2})(k + b\sqrt{2}) = -(a - c\sqrt{2})(a + c\sqrt{2})$$

which can be rearranged to

$$k + b\sqrt{2} = \left(\frac{r}{t} + \frac{s}{t}\sqrt{2}\right)(a + c\sqrt{2})$$

where $r = 2cb - ak$, $s = ck - ab$, $t = k^2 - 2b^2$. Furthermore, it is easy to verify (see [BUC89 pg. 40]) that the norm of the factor $\left(\frac{r}{t} + \frac{s}{t}\sqrt{2}\right)$ is -1 so that $k+b\sqrt{2}$ is simply a unit times $a+c\sqrt{2}$. Since the same is true for the other two equations in (1) we have

$$\begin{aligned} k + b\sqrt{2} &= u_1(a + c\sqrt{2}) \\ l + c\sqrt{2} &= u_2(b + a\sqrt{2}) \\ m + a\sqrt{2} &= u_3(c + b\sqrt{2}) \end{aligned}$$

where $u_i = (r_i + s_i\sqrt{2})/t_i$ and $\text{Norm}(u_i) = -1$. Equating the irrational parts leads to

$$\begin{aligned} r_1 c + s_1 a &= t_1 b \\ r_2 a + s_2 b &= t_2 c \\ r_3 b + s_3 c &= t_3 a. \end{aligned}$$

Solving the first two of these three for the ratios a/b and c/b gives

$$\frac{a}{b} = \frac{t_1 t_2 - r_1 s_2}{s_1 t_2 + r_1 r_2} \qquad \text{and} \qquad \frac{c}{b} = \frac{t_1 r_2 + s_1 s_2}{s_1 t_2 + r_1 r_2}. \tag{2}$$

The norm equations for the units imply the existence of integers p_i, q_i so that

$$\begin{aligned} r_i &= 2p_i q_i + p_i^2 - q_i^2 \\ s_i &= p_i^2 + q_i^2 \\ t_i &= \pm(2p_i q_i - p_i^2 + q_i^2). \end{aligned} \tag{3}$$

Finally, one substitutes the unit parametrizations (3) into the ratio equations (2) and defines $\phi := p_1/q_1$ and $\theta := p_2/q_2$ to obtain a parametrization of all[1] rational sided triangles with two rational medians, namely,

$$\begin{aligned} a &= \tau\{(-2\phi\theta^2 - \phi^2\theta) + (2\phi\theta - \phi^2) + \theta + 1\} \\ b &= \tau\{(\phi\theta^2 + 2\phi^2\theta) + (2\phi\theta - \theta^2) - \phi + 1\} \\ c &= \tau\{(\phi\theta^2 - \phi^2\theta) + (\theta^2 + 2\phi\theta + \phi^2) + \theta - \phi\}. \end{aligned}$$

Note that τ, ϕ, θ are rationals such that $\tau > 0$, $0 < \theta, \phi < 1$ and $\phi + 2\theta > 1$. These inequalities are obtained from the triangle inequality applied to the sides a, b and c. The medians to sides a and b are automatically rational while the median to side c, given by $m = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$, is not necessarily rational.

One can produce a list of triangles with three rational medians by simply enumerating over rational θ and ϕ and testing the third median. Since each such

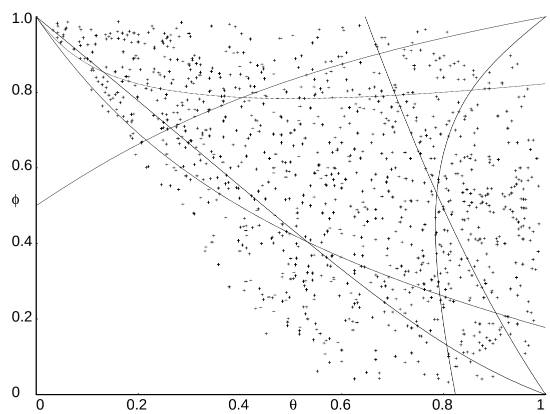

Figure 1: The points correspond to rational-sided triangles with 3 rational medians. The sixfold symmetry centred on the point $(\theta, \phi) = (\sqrt{3}/3, \sqrt{3}/3)$ is a result of the six permutations of the triangles sides (a, b, c). The six curves on the plot represent Euler's parametrization of a subset of these triangles.

triangle corresponds to a pair of rational numbers θ, ϕ, which one can think of as coordinates, it is possible to plot them in the $\theta\phi$-plane. The result (see Figure 1) shows some structure which is simply related to the fact that there are 6 points corresponding to each triangle, one for each permutation of the sides. However, the implication of the theorem above is that plotting all the triangles results in a dense picture since each fibre of $E[\phi](\mathbb{Q})$, which corresponds to a horizontal slice of this plot, contains an infinite number of points.

[1] The seeming loss of generality can be easily explained either by appeal to symmetry or casework (see [BUC89 pg. 41]).

Since τ is just a scaling factor which produces all triangles in a similarity class, we can, without loss of generality, take $\tau = 2$. Then expanding m in terms of θ and ϕ leads to

$$m^2 = c_4\theta^4 + c_3\theta^3 + c_2\theta^2 + c_1\theta + c_0$$

where

$$\begin{aligned}
c_4 &= (3\phi - 1)^2 \\
c_3 &= 2(9\phi^3 - 9\phi^2 - 11\phi - 1) \\
c_2 &= 3(3\phi^4 + 6\phi^3 + 2\phi^2 + 2\phi - 1) \\
c_1 &= 2(\phi - 1)(3\phi^3 - 8\phi^2 - 11\phi - 2) \\
c_0 &= (\phi - 1)^2(\phi + 2)^2.
\end{aligned}$$

Notice that m^2 is multiquartic in θ and ϕ (*ie.* quartic in each variable separately considering the remaining ones constant) and so if we fix ϕ say to some rational value ($\phi \neq 1/3$) then we can transform this equation into an elliptic curve. (We treat the case $\phi = 1/3$ separately by showing that it is isomorphic to the same family of elliptic curves.)

First we multiply by $(3\phi - 1)^6$ and set $U = (3\phi - 1)^2\theta$ and $V = (3\phi - 1)^3 m$ to obtain the monic quartic

$$V^2 = U^4 + c_3U^3 + (3\phi - 1)^2 c_2 U^2 + (3\phi - 1)^4 c_1 U + (3\phi - 1)^6 c_0.$$

Next we remove the cubic term with the usual transformation, namely $U = W - c_3/4$ which leads to the quartic

$$V^2 = W^4 + fW^2 + gW + h$$

where f, g and h are degree 6, 8 and 12 polynomials in ϕ, respectively. Finally, we use Mordell's bi-rational transformation of a quartic to a cubic [MOR 69, p. 139], namely $2W = (t - g/4)/(s + f/6)$, $V = 2s - W^2 - f/6$ which leads to

$$t^2 = 4s^3 - g_2 s + g_3$$

where

$$\begin{aligned}
g_2 &= \frac{9}{4}(3\phi^8 - 76\phi^6 - 110\phi^4 + 76\phi^2 + 3)(3\phi - 1)^4 \\
g_3 &= \frac{27}{8}(\phi^8 + 36\phi^6 - 58\phi^4 + 36\phi^2 + 1)(3\phi - 1)^6(\phi^2 + 1)^2.
\end{aligned}$$

To convert this to Weierstrass form set $s = (X + 3(3\phi - 1)^2(\phi^2 + 1)^2))/4$ and $t = Y/4$, then

$$Y^2 = X^3 + b_2^2 X^2 - b_4^2 X$$

where $b_2 = 3(3\phi - 1)(\phi^2 + 1)$ and $b_4 = 24\phi(\phi^2 - 1)(3\phi - 1)^2$. Now since $\phi \neq 1/3$ we can simplify this by using the transformation $X = 3^2(3\phi - 1)^2 x$ and $Y = 3^3(3\phi - 1)^3 y$ to obtain

$$E[\phi] : y^2 = x^3 + a_2^2 x^2 - a_4^2 x$$

where $a_2 = \phi^2 + 1$ and $a_4 = 8\phi(\phi^2 - 1)/3$.

The case $\phi = 1/3$ in the multiquartic equation for m^2 leads to

$$(9m/2)^2 = -(3\theta + 1)(72\theta^2 - 27\theta - 49).$$

Setting $\theta = -(81x/32 + 1)/3$ and $m = 81y/32$ converts this to

$$y^2 = x^3 + \left(\frac{10}{9}\right)^2 x^2 - \left(\frac{64}{81}\right)^2 x$$

which is precisely the same as $E[1/3]$. Clearly, all the above transformations have established the following theorem.

Theorem 2 *All rational triangles with three rational medians correspond to points on the elliptic curve*

$$E[\phi] : y^2 = x(x^2 + (\phi^2 + 1)^2 x - (8\phi(\phi^2 - 1)/3)^2)$$

where $\phi \in \mathbb{Q}$.

3 Torsion subgroup of $E[\phi](\mathbb{Q})$

We now show that the torsion subgroup of $E[\phi](\mathbb{Q})$ is just $\mathbb{Z}/2\mathbb{Z}$ by considering points of order $2, 3, 4$ and 5 and then using Mazur's theorem. If we fix on some particular rational value of ϕ in the region $0 < \phi < 1$ then the corresponding fibre on the surface of $E[\phi]$ has a discriminant given by

$$\Delta = \phi^4(\phi - 1)^4(\phi + 1)^4(9\phi^4 - 14\phi^2 + 9)(\phi^4 + 34\phi^2 + 1)$$

which is never zero (in the specified region). So $E[\phi]$ is non-singular.

3.1 Order 2 points

First notice that the point $(x, y) = (0, 0)$ is an order 2 point on $E[\phi]$. Any other point of order 2 satisfies $y = 0$ and $x \in \mathbb{Q}$ thus we require $x = 0$ or

$$x = \frac{-a_2^2 \pm \sqrt{(a_2^4 + 4a_4^2)}}{2} \in \mathbb{Q}$$

The latter is true if and only if the discriminant is a rational square so we require

$$(9\phi^4 - 14\phi^2 + 9)(\phi^4 + 34\phi^2 + 1) \in \mathbb{Q}^2.$$

Now consider the problem of finding all rational points on

$$Y^2 = X^4 + 34X^2 + 1$$

which can be transformed (again via Mordell) into the elliptic curve

$$y^2 = x(x - 8)(x - 9).$$

One can (see [BUC'89 p.65] or via `apecs` routines) show that this last curve has rank 0 and that the torsion subgroup is just

$$\{\mathcal{O}, (0,0), (8,0), (9,0), (6,\pm 6), (12,\pm 12)\}$$

which implies that the only way $\phi^4 + 34\phi^2 + 1$ can be a rational square is when $\phi = 0, \pm 1$. Similarly, the curve $Y^2 = 9X^4 - 14X^2 + 9$ transforms to the same elliptic curve and so the only way $9\phi^4 - 14\phi^2 + 9$ can be a rational square is when $\phi = 0, \pm 1$. So in the region defined by $0 < \theta, \phi < 1$ neither factor of the discriminant can be individually a rational square.

Now set $\phi = p/q$ where $\gcd(p,q) = 1$. The condition that the discriminant be a rational square becomes

$$(9p^4 - 14p^2q^2 + 9q^4)(p^4 + 34p^2q^2 + q^4) = r^2$$

for some integer r. Let $\gcd(9p^4 - 14p^2q^2 + 9q^4, p^4 + 34p^2q^2 + q^4) = d$ then $d \mid 160 \gcd(-2p^2q^2, p^4 + q^4)$ implies that $d \mid 320$ since $\gcd(-2p^2q^2, p^4 + q^4) \mid 2$. So we can say

$$\begin{aligned} 9p^4 - 14p^2q^2 + 9q^4 &= dU \\ p^4 + 34p^2q^2 + q^4 &= dV \end{aligned}$$

where $\gcd(U,V) = 1$. Substituting this into the discriminant condition above implies that $UV = \Box$ hence U and V are perfect squares. So let $U = u^2$ and $V = v^2$ which means we need to solve the pair

$$\begin{aligned} 9p^4 - 14p^2q^2 + 9q^4 &= du^2 \\ p^4 + 34p^2q^2 + q^4 &= dv^2 \end{aligned}$$

for any d dividing 320. Since we can consider d to be squarefree, without loss of generality, it is sufficient to just let $d = \pm 1, \pm 2, \pm 5, \pm 10$.

If $d < 0$ the second equation has no non-trivial real solutions since the left hand side is greater than zero while the right hand side is less than or equal to zero. Thus the pair have no non-trivial integer solutions in this case.

If $d = 1$ then both equations reduce to the elliptic curve $y^2 = x(x-8)(x-9)$ as before and so have no non-trivial solutions.

If $d = 2$ then the second equation taken modulo 8 becomes

$$(p^2 + q^2)^2 \equiv 2v^2 \pmod 8.$$

Thus $p^2 + q^2 \equiv 0, 4 \pmod 8$ implies that $p, q \equiv 0 \pmod 2$ which contradicts $\gcd(p,q) = 1$. So there are no solutions in this case.

If $d = 5$ we again consider the second equation modulo 5 this time which becomes

$$(p^2 + q^2)^2 \equiv 3p^2q^2 \pmod 5.$$

Thus $p^2 + q^2 \equiv 0 \pmod 5$ and $pq \equiv 0 \pmod 5$ which together imply that 5 divides both p and q again contradicting $\gcd(p,q) = 1$.

If $d = 10$ the second equation leads to

$$(p^2 + q^2)^2 \equiv 2v^2 \pmod 8$$

which implies there are no solutions by the same argument as for the case $d = 2$.

So the only 2-torsion point is $(x, y) = (0, 0)$. By Mazur's theorem we conclude that the torsion subgroup can only be of the form $\mathbb{Z}/n\mathbb{Z}$ for $n = 2, 4, 6, 8, 10, 12$.

3.2 Order 3 points

Suppose $P = (x, y)$ is an order 3 point on $E[\phi]$ then we require $2P = -P$. Letting $x(2P)$ and $y(2P)$ denote the x and y coordinates of the point $2P$ respectively, we find that

$$x(2P) = (x^2 + a_4^2)^2/4y^2$$
$$y(2P) = p_6(x)/8y^3$$

where $p_6(x) = -(x^2+a_4^2)(x^4+2a_2^2x^3-6a_4^2x^2-2a_2^2a_4^2x+a_4^4)$. Now the requirement that $x(2P) = x$ leads to the quartic condition

$$f(x, a_2, a_4) = 3x^4 + 4a_2^2x^3 - 6a_4^2x^2 - a_4^4 = 0$$

while the y-coordinate, $y(2P) = -y$, leads to the sextic condition

$$g(x, a_2, a_4) = p_6(x) + 8y^4 = 0.$$

To find the common zeros of f and g we calculate the resultant of the two polynomials $Res(f, g, x)$ relative to the variable x. This results in

$$Res(f, g, x) = 2^{16}a_4^{16}(a_2^4 + 4a_4^2)^4$$

which is zero if and only if $a_4 = 0$ or $a_4 = \pm a_2^2 i/2$.

The case $a_4 = 0$ leads to

$$f(x, a_2, 0) = x^3(3x + 4a_2^2)$$
$$g(x, a_2, 0) = x^4(7x^2 + 14a_2^2x + 8a_2^4)$$

which have common zeros when $x = 0$ or when

$$Res\left(\frac{f(x, a_2, 0)}{x^3}, \frac{g(x, a_2, 0)}{x^3}, x\right) = -64a_2^6 = 0.$$

However $f(x, 0, 0) = 3x^4$ and $g(x, 0, 0) = 7x^6$. So f and g have common zeros when $(x, a_2, a_4) = (0, r, 0)$ for $r \in \mathbb{Q}$.

Recall that a_2 and a_4 are rational so the case $a_4 = \pm a_2^2 i/2$ cannot lead to any solutions.

We conclude that there are no order 3 points which implies that the torsion subgroup is restricted to just $\mathbb{Z}/n\mathbb{Z}$ for $n = 2, 4, 8, 10$.

3.3 Order 4 points

If $P = (x, y)$ is an order 4 point on $E[\phi]$ then we require $2P = (0, 0)$ since that is the only order 2 point. Using the expressions for $2P$ above leads to

$$x(2P) = 0 \iff (x^2 + a_4^2)^2 = 0$$

whence $x = \pm a_4 i$. So there are no order 4 points on $E[\phi]$ which means that the torsion subgroup is restricted to just $\mathbb{Z}/n\mathbb{Z}$ for $n = 2, 10$.

3.4 Order 5 points

To decide whether the torsion subgroup is $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/10\mathbb{Z}$ requires consideration of order 5 points. Suppose $P = (x, y)$ is an order 5 point on $E[\phi]$. This is equivalent to $4P = -P$. Recall that

$$x(2P) = (x^2 + a_4^2)^2/4y^2$$
$$y(2P) = p_6(x)/8y^3$$

so that a recursive substitution gives $4P$ in terms of x and y as

$$x(4P) = \left\{\frac{(x^2 + a_4^2)^4 + 16y^4a_4^2}{4yp_6(x)}\right\}^2$$
$$y(4P) = \frac{P_6((x^2 + a_4^2)^2, y^2)}{64p_6^3(x)y^3}$$

where $P_6(X, Y)$ is just the homogenization of $p_6(x)$ namely

$$P_6(X, Y) = -(X^2 + a_4^2Y^2)(X^4 + 2a_2^2X^3Y - 6a_4^2X^2Y^2 - 2a_2^2a_4^2XY^3 + a_4^4Y^4).$$

Now, as in the case of order 3 points, we obtain two conditions on x

$$x(4P) = x \iff f_{16}(x) = 0$$

$$y(4P) = -y \iff g_{25}(x) = 0$$

where the degree 16 and 25 polynomials in x are given by

$$f_{16}(x, a_2, a_4) = \{(x^2 + a_4^2)^4 + 16y^4a_4^2\}^2 - 16xy^2p_6^2(x)$$
$$g_{25}(x, a_2, a_4) = P_6((x^2 + a_4^2)^2, y^2) + 64p_6^3(x)y^4.$$

The resultant of these two polynomials is

$$Res(f_{16}, g_{25}, x) = 2^{240}a_4^{256}(a_2^4 + 4a_4^2)(a_4^4 + 6a_4^2 + 4a_2^2 - 3)p_{12}(a_2, a_4)$$

where

$$\begin{aligned} p_{12}(a_2, a_4) = a_4^{12} &+ 50a_4^{10} - (140a_2^2 + 125)a_4^8 + (160a_2^4 + 368a_2^2 + 300)a_4^6 \\ &- (64a_2^6 + 240a_2^4 + 360a_2^2 + 105)a_4^4 - (80a_2^2 + 62)a_4^2 \\ &+ (16a_2^4 + 20a_2^2 + 5). \end{aligned}$$

Setting the resultant to zero provides us with the following cases for common roots.

i. $a_4 = 0$,

ii. $a_2^4 + 4a_4^2 = 0$,

iii. $a_4^4 + 6a_4^2 + 4a_2^2 - 3 = 0$ or

iv. $p_{12}(a_2, a_4) = 0$.

Just as in the analysis of the order 3 points we find that conditions i. and ii. provide no solutions and hence no order 5 points.

Case iii. If we consider this as a quadratic in a_4^2 then the condition that $a_2, a_4 \in \mathbb{Q}$ means that the discriminant is a rational square. Thus we require $48 - 16a_2^2 = k^2$ for some $k \in \mathbb{Q}$. However, using a modulo 3 argument shows that the only solution to this is $a_2 = 0$. So we have $(a_2, a_4) = (0, 0)$ or $(0, \sqrt{6}i)$ both of which fail to produce order 5 points.

Case iv. Recall that $a_2 = \phi^2+1$ and $a_4 = 8\phi(\phi^2-1)/3$ so that this condition becomes

$$p_{12}(\phi^2 + 1, 8\phi(\phi^2 - 1)/3) = p_{36}(\phi) = 0$$

where $p_{36}(\phi)$ is irreducible over $\mathbb{Q}$.

So we are left with the result that there are no order 5 points which means that the torsion group is $\mathbb{Z}/2\mathbb{Z}$.

4 Rank of $E[\phi](\mathbb{Q})$

To obtain a lower bound for the rank we use the fact that $E[\phi](\mathbb{Q})$ always has an order 2 point and so the usual 2-descent homomorphism, α say, from $E[\phi](\mathbb{Q})$ into $\mathbb{Q}^*/\mathbb{Q}^{*2}$ can be applied.

First we let $\phi = p/q$ so that $E[\phi]$ is isomorphic to

$$E[p, q] : y^2 = x(x^2 + a^2x - b^2)$$

where $a = 3(p^2 + q^2)$ and $b = 24pq(p^2 - q^2)$. Then the homogeneous spaces corresponding to $E[p, q]$ are obtained by setting $x = dr^2/s^2$ and $y = drt/s^3$ whence

$$dt^2 = d^2r^4 + da^2r^2s^2 - b^2s^4$$

for any squarefree d which divides $6pq(p-q)(p+q)$. For any choice of relatively prime p and q we see that there will always be divisors $d = \pm1, \pm2, \pm3, \pm6$. A systematic search led to general solutions in four of these cases, namely,

$$\begin{aligned} d = 1 &: (r, s, t) = (8pq, 1, 80p^2q^2) \\ d = -1 &: (r, s, t) = (6pq, 1, 30pq(p^2 - q^2)) \\ d = 2 &: (r, s, t) = (6p(q - p), 1, 6p(p - q)(9p^2 - 8pq + q^2)) \\ d = -2 &: (r, s, t) = (2p(q - p), 1, 2p(p - q)(p^2 + 8pq + 9q^2)). \end{aligned}$$

The way these were obtained was to fix $s = 1$ (for each value of d) and then inspect a list of solutions $(r, 1, t)$ for various values of the parameters p and q. For example, one can fix $p = 1$ and observe how the solutions $(r, 1, t)$ vary with q. Then do the same for $p = 2, 3, 4, ...$, etc., until one can observe the p-dependence. Essentially, there is nothing more than pattern recognition and polynomial fitting going on here. So the image of α contains at least 4 points, *ie.* $|\alpha(E[\phi])| \geq 4$.

Secondly, the 2-isogenous curve to $E[p,q]$ is given [SIL 86, p.302] by

$$\overline{E}[p,q] : Y^2 = X(X^2 + \overline{a}X + \overline{b})$$

where $\overline{a} = -2a^2$ and $\overline{b} = a^4 + 4b^2$. As above, we set $X = DR^2/S^2$ and $Y = U/V$ to obtain the homogeneous spaces to $\overline{E}[p,q]$, namely,

$$DT^2 = D^2R^4 + D\overline{a}R^2S^2 + \overline{b}S^4$$

such that D is a squarefree divisor of $\overline{b}$. Now we find that

$$\overline{b} = a^4 + 4b^2 = 9(9p^4 - 14p^2q^2 + 9q^4)(p^4 + 34p^2q^2 + q^4).$$

As before, a tedious search reveals three solutions, namely,

$$\begin{aligned} D = D_1 = 1 &: (R,S,T) = (10pq, 1, 9p^4 + 46p^2q^2 + 9q^4) \\ D = D_2 = p^4 + 34p^2q^2 + q^4 &: (R,S,T) = (1, 1, 8(p^2 - q^2)) \\ D = D_3 = 9p^4 - 14p^2q^2 + 9q^4 &: (R,S,T) = (1, 1, 16pq). \end{aligned}$$

Now there are a number of potential problems with these three solutions.

i. The last two values could be squares and so would not produce a solution distinct to $D = 1$. Recall that in Section 3.1 we showed that these factors could not be perfect squares so they must contain some non-trivial squarefree part.

ii. The squarefree parts of D_2 and D_3 might be the same in which case they do not produce distinct solutions. However, they must be distinct otherwise we would conclude that the product, D_2D_3, is a perfect square again contradicting the results of Section 3.1.

iii. It is possible that a solution for a non-squarefree D does not lead to a solution for a D equal to the squarefree part. Suppose we have a solution (R,S,T) of the homogeneous space when $D = k^2\delta$ say for some squarefree δ and non-trivial k. Then we find that (kR, S, kT) is a solution to

$$\delta T^2 = \delta^2R^4 + \delta\overline{a}R^2S^2 + \overline{b}S^4$$

which is the same equation but with a squarefree δ in place of D.

With these problems out of the way we continue the rank calculation. Since the image of $\alpha(\overline{E}[p,q])$ must be a power of two we can say that it is at least 4, *ie.* $|\alpha(\overline{E}[p,q])| \geq 4$.

The rank, $\mathrm{r}(E)$, of $E[p,q]$ is given [S-T 92, pp.89-98] by

$$2^{r(E)} = \frac{|\alpha(E)|\,|\alpha(\overline{E})|}{4}$$

we see that the rank of $E[p,q]$ is at least 2.

Having proven that the rank is never zero means that each fibre of $E[\phi]$ contains an infinite number of points. However to check that Figure 1 really does end up being a dense plot of points requires that this be so in the region defined by $0 < \theta, \phi < 1$ and $\phi + 2\theta > 1$. Suppose we concentrate on the fibre $E[{}^1\!/\!_2]$. Using the transformations of Section 2 we find the inequalities

$$\frac{1}{4} < \theta < 1,$$

$$\frac{1}{16} < U < \frac{1}{4},$$

$$-\frac{60}{16} < W < -\frac{57}{16},$$

$$-\frac{60}{8} < \frac{t - g/4}{s + f/6} < -\frac{57}{8}$$

are equivalent. Now from the last one we see that if the denominator is positive then

$$-\frac{60}{8}(s + f/6) < -\frac{57}{8}(s + f/6)$$

or $s > f/6$. Alternatively, if the denominator is negative then

$$-\frac{60}{8}(s + f/6) > -\frac{57}{8}(s + f/6)$$

which implies that $s < f/6$. In other words, there are no restrictions on the domain of s on the curve $t^2 = 4s^3 - g_2 s + g_3$. Since this last curve contains an infinite number of rational points this implies that there are an infinite number of points in the region $\frac{1}{4} < \theta < 1$ on the curve $E[{}^1\!/\!_2]$. A similar argument works for any fibre and in fact for any subregion so that the limit as all triangles are plotted looks something like Figure 2.

5 Computational Results

If we set $\phi = {}^p\!/\!_q$ in the elliptic curve

$$E[\phi] : y^2 = x(x^2 + (\phi^2 + 1)^2 x - (8\phi(\phi^2 - 1)/3)^2)$$

then a simple transformation reveals that this is isomorphic to

$$E[p,q] : y^2 = x(x^2 + (p^2 + q^2)^2 x - (8pq(p^2 - q^2)/3)^2)$$

Notice that $E[p,q]$ is symmetric in p and q so that it is sufficient to consider $1 \leq p < q$ and $\gcd(p,q) = 1$. Furthermore, the transformation $(p,q) \mapsto (q-p, q+p)$ leads to an isomorphic curve so we can restrict attention to those p,q pairs for which $2 \mid \gcd(p,q)$ without loss of generality.

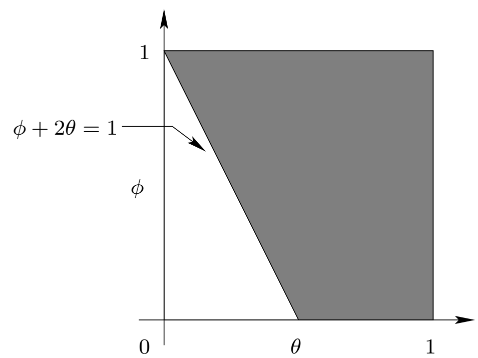

Figure 2: The limiting result of plotting all triangles with three rational medians

Using both Cannon's `Magma` package and Cremona's `mwrank` program running on a 450 Mhz Celeron under `Linux` one can readily determine the ranks of the curves $E[p, q]$ in the range $1 \leq p \leq q \leq 100$ as shown in the tables in the Appendix.

An extract showing the p, q pair with the smallest sum for a given rank is displayed in Table 1. The curve $E[17, 70]$, for example, has rank 7 which was

p	q	$p+q$	rank($E[p,q]$)
1	2	3	2
1	4	5	3
2	9	11	4
8	9	17	5
17	24	41	6
17	70	87	7

Table 1: Rank of some curves $E[p, q]$

verified via Connell's `apecs`. It is somewhat surprising that such reasonably large rank curves emerge from a problem not explicitly designed to create high rank curves. Notice also the rough doubling in size of $p + q$ for each increase in the rank by 1.

6 Acknowledgement

Much of this work would not have been possible without the support of DSTO and the advanced facilities they supply. Also the package `maple` along with the elliptic curve add-on `apecs` proved invaluable for much of the analysis. Independent work by George Cole [Col'91] on intersections of two quadric surfaces led to a number of similar results.

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8 Appendix

Table A.1 rank$(E[p,q](\mathbb{Q}))$

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p\q|   2   4   6   8  10  12  14  16  18  20  22  24  26  28  30  32  34  36  38  40  42  44  46  48  50
---|----------------------------------------------------------------------------------------------------
  1| . 2 . 3 . 3 . 3 . 3 . 3 . 4 . 4 . 4 . 4 . 3 . 5 . 4 . 5 . 5 . 5 . 5 . 3 . 5 . 4 . 5 . 4 . 3 . 4 . 4
  2| . . 3 . 3 . 3 . 4 . 4 . 4 . 3 . 5 . 4 . 3 . 4 . 4 . 5 . 5 . 4 . 3 . 4 . 4 . 5 . 3 . 4 . 4 . 4 . 4 .
  3| . . . 3 . . . 3 . 4 . . . 3 . 4 . . . 4 . 4 . . . 4 . 4 . . . 3 . 4 . . . 4 . 4 . . . 3 . 4 . . . 4
  4| . . . . 3 . 3 . 3 . 4 . 3 . 4 . 3 . 3 . 4 . 4 . 5 . 4 . 4 . 3 . 5 . 4 . 4 . 4 . 3 . 4 . 4 . 3 . 3 .
  5| . . . . . 3 . 4 . . . 4 . 3 . 3 . 4 . . . 4 . 3 . 3 . 5 . . . 5 . 4 . 4 . 4 . . . 3 . 3 . 5 . 5 . .
  6| . . . . . . 3 . . . 4 . 3 . . . 4 . 4 . . . 4 . 3 . . . 3 . 4 . . . 3 . 4 . . . 4 . 4 . . . 5 . 4 .
  7| . . . . . . . 3 . 4 . 4 . . . 4 . 3 . 4 . 4 . 5 . 4 . . . 4 . 4 . 4 . 4 . 5 . 3 . . . 5 . 4 . 4 . 5
  8| . . . . . . . . 5 . 4 . 3 . 4 . 4 . 4 . 4 . 5 . 3 . 4 . 4 . 4 . 3 . 4 . 4 . 4 . 5 . 4 . 3 . 4 . 4 .
  9| . . . . . . . . . 3 . . . 4 . 4 . . . 4 . 4 . . . 3 . 5 . . . 4 . 5 . . . 4 . 3 . . . 3 . 5 . . . 4
 10| . . . . . . . . . . 3 . 4 . . . 3 . 5 . 4 . 4 . . . 3 . 3 . 5 . 5 . . . 4 . 4 . 4 . 3 . . . 4 . 4 .
 11| . . . . . . . . . . . 3 . 4 . 4 . 3 . 3 . . . 5 . 4 . 4 . 5 . 3 . 4 . 3 . 3 . 5 . 5 . . . 5 . 4 . 4
 12| . . . . . . . . . . . . 3 . . . 3 . 4 . . . 3 . 3 . . . 3 . 4 . . . 4 . 4 . . . 5 . 4 . . . 3 . 4 .
 13| . . . . . . . . . . . . . 3 . 4 . 4 . 4 . 4 . 3 . . . 5 . 4 . 4 . 3 . 4 . 5 . 5 . 4 . 4 . 5 . 4 . 3
 14| . . . . . . . . . . . . . . 4 . 3 . 5 . . . 4 . 3 . 4 . 5 . 4 . 3 . . . 3 . 4 . 5 . 3 . 3 . 4 . . .
 15| . . . . . . . . . . . . . . . 3 . . . . . 3 . . . 4 . 5 . . . 5 . 4 . . . 4 . . . . . 4 . 4 . . . .
 16| . . . . . . . . . . . . . . . . 4 . 4 . 5 . 3 . 4 . 5 . 4 . 4 . 4 . 3 . 4 . 4 . 3 . 5 . 4 . 4 . 4 .
 17| . . . . . . . . . . . . . . . . . 5 . 4 . 4 . 6 . 3 . 3 . 4 . 4 . . . 4 . 3 . 4 . 4 . 5 . 4 . 4 . 5
 18| . . . . . . . . . . . . . . . . . . 4 . . . 4 . 3 . . . 4 . 3 . . . 4 . 4 . . . 3 . 3 . . . 4 . 5 .
 19| . . . . . . . . . . . . . . . . . . . 4 . 4 . 4 . 4 . 5 . 3 . 4 . 4 . 5 . . . 3 . 4 . 4 . 4 . 4 . 4
 20| . . . . . . . . . . . . . . . . . . . . 4 . 5 . . . 3 . 4 . 4 . 4 . . . 4 . 4 . 4 . 4 . . . 4 . 4 .
 21| . . . . . . . . . . . . . . . . . . . . . 4 . . . 4 . . . . . 4 . 3 . . . 5 . 4 . . . 4 . 5 . . . 3
 22| . . . . . . . . . . . . . . . . . . . . . . 5 . 3 . 3 . 4 . 4 . . . 4 . 4 . 4 . 4 . 3 . 4 . 4 . 4 .
 23| . . . . . . . . . . . . . . . . . . . . . . . 4 . 5 . 3 . 4 . 4 . 4 . 3 . 4 . 4 . 4 . 4 . . . 5 . 3
 24| . . . . . . . . . . . . . . . . . . . . . . . . 3 . . . 4 . 3 . . . 4 . 4 . . . 4 . 5 . . . 3 . 3 .
 25| . . . . . . . . . . . . . . . . . . . . . . . . . 3 . 3 . . . 3 . 4 . 3 . 4 . . . 3 . 4 . 3 . 4 . .
 26| . . . . . . . . . . . . . . . . . . . . . . . . . . 4 . 5 . 5 . 4 . 4 . 3 . . .(3). 4 . 3 . 4 . 5 .
 27| . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 . . . 5 . 5 . . . 4 . 4 . . . 3 . 4 . . . 3
 28| . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5 . 4 . 4 . . . 4 . 3 . 5 . 3 . 3 . 4 . . .
 29| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3 . 4 . 4 . 3 . 3 . 4 . 3 . 3 . 6 . 4 . 4
 30| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 . . . . . 4 . . . 4 . 3 . . . 3 . 4 .
 31| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 . 4 .(3). 4 . 5 . 3 . 4 . 5 . 4 . 3
 32| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3 . 3 . 5 . 3 . 4 . 4 . 3 . 3 . 4 .
 33| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6 . . . 4 . 4 . . . . .(3). . . 5
 34| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 . 4 . 3 . 3 . 4 . 4 . 4 . 4 .
 35| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3 . 4 . . . . . 4 . 5 . 3 . .
 36| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3 . . . 4 . 4 . . . 4 . 5 .
 37| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3 . 4 . 5 . 5 . 3 . 3 . 3
 38| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 . 4 . 3 . 5 . 4 . 6 .
 39| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3 . . . 4 . 5 . . . 4
 40| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 . 3 . . . 3 . 4 .
 41| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3 . 4 . 5 . 3 . 3
 42| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 . . . 4 . . .
 43| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 . 4 . 4 . 5
 44| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5 . 4 . 4 .
 45| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 . . . .
 46| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 . 3 .
 47| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3 . 3
 48| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5 .
 49| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4
 50| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
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Note that those ranks contained in parentheses are lower bounds.

Table A.2 $\mathrm{rank}(E[p,q](\mathbb{Q}))$

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---|----------------------------------------------------------------------------------------------------
p\q|  52  54  56  58  60  62  64  66  68  70  72  74  76  78  80  82  84  86  88  90  92  94  96  98 100
---|----------------------------------------------------------------------------------------------------
  1| . 3 . 3 . 3 . 5 . 5 . 4 . 3 . 4 . 4 . 4 .(3). 4 . 4 . 5 . 3 . 4 . 3 . 5 .(3). 4 . 4 . 5 . 3 . 5 . 4
  2| 3 . 3 . 4 . 3 . 4 . 3 . 4 . 4 . 3 .(3). 4 . 4 . 4 . 3 . 5 . 4 . 3 . 4 . 4 . 4 . 4 . 5 . 4 . 3 . 3 .
  3| . 4 . . . 4 . 3 . . . 5 . 4 . . . 4 . 4 . . . 5 . 4 . . . 3 . 4 . . . 4 . 4 . . . 3 . 4 . . . 4 . 5
  4| 4 . 3 . 4 . 5 . 4 . 4 . 3 . 4 . 3 . 3 . 4 . 3 . 5 . 6 . 4 . 6 . 4 . 4 . 4 . 5 . 5 . 3 . 4 . 4 . 4 .
  5| . 5 . 5 . 5 . 5 . . . 4 . 4 . 3 . 4 . . . 4 . 3 . 3 . 5 . . . 4 . 3 . 5 . 3 . . . 5 . 5 . 4 . 4 . .
  6| . . 4 . 4 . . . 5 . 5 . . . 5 . 3 . . . 3 . 3 . . .(3). 5 . . . 4 . 3 . . . 4 . 4 . . . 5 . 4 . . .
  7| . 3 . 4 . . . 5 . 4 . 4 . 4 . 5 . 3 . . .(3). 4 . 3 . 6 . 5 . 4 . . . 4 . 3 . 3 . 4 . 4 . 5 . . . 3
  8| 6 . 4 . 3 . 4 . 3 . 4 . 4 . 3 . 6 . 4 .(3). 4 . 3 . 4 . 5 . 6 . 3 . 5 . 3 . 4 . 3 . 5 . 4 . 6 . 3 .
  9| . 5 . . . 4 . 5 . . . 4 . 4 . . . 4 .(3). . . 4 . 5 . . . 5 .(3). . . 3 . 4 . . . 3 . 5 . . . 4 . 4
 10| 3 . 5 . . . 4 . 3 . 5 . 3 . . . 4 . 4 . 3 . 4 . . . 3 . 4 . 4 . 5 . . . 4 . 5 . 4 .(3). . . 5 .(4).
 11| . 4 . 3 . 5 . 3 . 4 . 3 . 4 . . . 6 . 4 . 4 . 5 . 5 . 4 . 5 . 5 . 4 . 4 . . . 5 . 4 .(4). 4 . 4 . 3
 12| . . 5 . 5 . . . 5 . 3 . . . 4 . 4 . . . 4 . 4 . . . 3 . 5 . . . 4 . 4 . . . 5 . 3 . . . 4 . 3 . . .
 13| . . . 5 . 5 . 5 . 4 . 4 . 4 . 5 . 4 . 4 . 5 . 4 . 5 . . . 3 . 3 .(3). 3 . 4 . 4 . 5 . 3 . 4 . 4 . 3
 14| 5 . 3 . 4 . 4 . 4 . 4 . . . 6 . 4 .(3). 5 . 4 . 5 . . . 4 . 5 .(3). 4 . 4 . 5 . . . 4 . 4 . 4 . 4 .
 15| . 3 . . . 3 . 4 . . . 4 . 3 . . . 6 . . . . . 4 . 3 . . . . . 4 . . . 4 . 3 . . . 3 . 6 . . . 4 . .
 16| 4 . 4 . 4 . 5 . 3 . 3 . 3 . 4 . 3 . 3 . 5 . 6 . 3 . 4 . 5 . 3 . 4 . 4 . 4 . 4 . 4 . 4 .(3). 4 . 3 .
 17| . 5 . 5 . 3 . 5 . 4 . 3 . 4 . 4 . . . 7 . 5 . 3 . 4 . 4 . 4 . 4 . 4 . 4 . 3 . 5 . 5 .(3). 5 . 6 . 5
 18| . . 4 . 5 . . . 3 .(4). . . 4 . 4 . . . 5 . 4 . . . 3 . 4 . . . 4 . 3 . . . 5 .(3). . . 4 . 4 . . .
 19| . 4 . 4 . 5 . 3 . 5 . 5 . 5 . 4 . 3 . 3 . 5 . 3 . . . 4 . 5 . 5 . 3 . 4 . 5 . 5 . 4 . 4 . 4 . 4 . 4
 20| 4 . 4 . . . 6 . 4 . 4 . 4 . . . 3 . 4 . 3 . 4 . . . 3 . 5 . 5 . 4 . . . 3 . 4 . 4 . 5 . . . 4 . 3 .
 21| . 3 . . . . . 3 . . . 4 . 4 . . . 3 . . . . . 4 . 5 . . . 4 . 3 . . . 5 . 3 . . . 4 . 4 . . . . . 3
 22| 5 . 3 . . . 4 . 3 . 4 . 4 . 4 . 5 . 5 . 6 . 4 . 4 . . . 3 . 5 . 3 . 3 . 5 . 3 . 5 . 3 . 4 . 3 . . .
 23| . 4 . 3 . 5 . 4 . 5 . 4 . 5 . 5 . 4 . 5 . 4 . 3 . 4 . 4 . 3 . 4 . 5 . 5 . 4 . 4 . . . 4 . 4 . 4 . 4
 24| . . 3 . 4 . . . 4 . 4 . . . 5 . 4 . . . 6 . 5 . . . 6 . 3 . . . 4 . 3 . . . 5 . 3 . . . 3 . 4 . . .
 25| . 5 . 4 . 5 . 5 . . . 6 . 4 . 5 . 3 . . . 4 . 5 . 6 . 4 . . . 5 . 4 . 4 . 5 . . . 4 . 4 . 5 . 5 . .
 26| 4 . 4 . 4 . 5 . 5 . 4 . 4 . . . 5 . 4 . 4 . 4 . 4 . 4 . 4 . 4 . 5 . 3 . 3 . 3 . . . 5 . 3 . 6 . 3 .
 27| . 4 . . . 4 . 4 . . . 4 . 5 . . . 5 . 3 . . . 6 . 4 . . . 5 . 4 . . . 3 . 4 . . . 5 .(3). . . 5 . 5
 28| 4 . 3 . 3 . 4 . 4 . 4 . . . 5 . 5 . 4 . 3 . 4 . 4 . . .(3). 5 . 4 . 3 . 4 . 4 . . . 4 . 3 . 3 . 4 .
 29| . 5 . 4 . 5 . . . 4 . 3 . 4 . 5 . 5 . 3 . 3 . 5 . 5 . 3 . 3 . 4 . 3 . 5 . 3 . 4 . 3 . 6 . 5 . 4 . 3
 30| . . 4 . . . . . 5 . 3 . . . . . 5 . . . 4 . 3 . . . 4 . 5 . . .(3). . . . . 5 . 5 . . . . .(3). . .
 31| . 4 . 4 . 4 . 4 . 4 . . . 3 . 5 . 3 . 3 .(3). 4 . 3 . 4 . 3 . 3 . 4 . 4 . 4 . 3 .(3). 4 . 6 . 4 . 4
 32| 4 . 3 . 4 . 4 . 4 . 3 . 5 . 5 . 3 . 4 . 4 . 5 . 4 . 4 . 4 . 5 . 3 . 4 . 6 . 4 . 3 . 5 . 4 . 6 . 4 .
 33| . 5 . . . 3 . 4 . . . 5 . 3 . . . 3 .(3). . . 5 . 4 . . . 4 . 4 . . . 4 . . . . . 5 . 4 . . .(3). 5
 34| . . 5 . 4 . 5 . 5 . 4 . 5 . 4 . 3 .(3). 5 . 6 . 4 . 4 . 3 . 5 . 3 . . .(4). 5 . 4 . 3 . 5 . 5 . 4 .
 35| . 4 . 4 . . . 6 . . . 4 . 4 . 3 . 5 . . . 4 . 4 . 4 . 3 . . . 4 . . . 5 . 4 . . . 3 . 6 . 3 . . . .
 36| . . 5 . 5 . . . 3 . 4 . . . 4 . 4 . . . 4 . 4 . . . 4 . 4 . . . 5 . 3 . . . 4 . 4 . . . 3 . 5 . . .
 37| . 4 . 3 . 4 . 5 . 3 . 3 . 3 . 5 . 4 . 3 . 4 . . . 4 . 4 . 4 . 3 . 3 . 3 . 4 . 3 . 4 . 4 . 5 . 4 . 4
 38| 3 . 3 . 3 . . . 4 . 4 . 3 . 4 . 4 . 4 .(3). 3 . 4 . 5 . 4 . 3 . 4 . 4 . 5 . 5 . 4 . 5 . . . 4 . 4 .
 39| . . . . . 4 . 5 . . . 4 . 4 . . . 4 . 3 . . . 3 . 5 . . . 4 . 3 . . . 4 . 4 . . . 5 . 4 . . . 3 . 5
 40| 3 . 4 . . . 4 . 4 . 3 . 5 . . . 3 . 5 . 4 . 3 . . . 3 . 3 . 3 . 3 . . . 4 . 3 . 5 . 4 . . . 4 . 3 .
 41| . 5 . 4 . 5 . 3 . 5 . 4 . 4 . 4 . 4 . 5 . 4 . 4 . 4 . 4 .(3). . . 4 . 4 . 4 . 5 . 5 . 4 . 4 . 4 . 4
 42| . . 4 . 4 . . . 3 . 5 . . . 3 . 3 . . . 3 . 4 . . . . . 5 . . . 4 . 5 . . .(3). . . . . 3 . 4 . . .
 43| . 5 . 4 . 3 .(3). 5 . 4 . 3 . 4 . 5 . 5 . 5 . 3 . 5 . 5 . 3 . 4 . 4 . . . 3 . 4 . 3 . 5 . 4 . 3 . 3
 44| 4 . 3 . . .(3). 5 . 4 . 4 . 5 . 3 . 3 . 3 . 5 . 6 . . . 5 . 4 . 4 . 4 . 5 . 4 . 3 . 3 . 4 . 4 . . .
 45| . 5 . . . 3 . 3 . . . 4 . 3 . . . 4 . . . . . 4 . 5 . . . . . 4 . . . 3 . 5 . . . 4 . 5 . . . 4 . .
 46| 4 . 4 . 3 . 3 . 5 . 3 . 5 . 4 . 5 . . . 3 . 5 . 5 . 4 . 4 . 4 .(4). 3 . 4 . 4 . 4 . 4 . 5 . 4 . 4 .
 47| . 6 . 5 . 4 . 4 . 4 . 6 . 5 . 4 . 4 . 5 . 5 . 4 . 6 . 4 . 5 . 4 . 3 . 5 . 3 . 4 . 3 . . . 5 . 3 . 6
 48| . . 5 . 3 . . . 4 . 3 . . . 5 . 3 . . . 3 . 3 . . . 5 . 3 . . . 5 . 5 . . . 4 . 3 . . . 6 . 3 . . .
 49| . 4 . 4 . . . 5 . 3 . 5 . 4 . 3 . 4 . . . 4 . 4 . 5 . 6 . 5 . 5 . . . 3 . 4 . 4 . 4 . 4 . 5 . . . 6
 50| 6 . 5 . . . 4 . 4 .(3). 4 . . . 6 . 4 . 5 . 4 . . . 3 . 3 . 4 . 3 . . . 4 . 4 . 4 . 3 . . . 4 . 4 .
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Table A.3 rank($E[p,q](\mathbb{Q})$)

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---|----------------------------------------------------------------------------------------------------
p\q|  52  54  56  58  60  62  64  66  68  70  72  74  76  78  80  82  84  86  88  90  92  94  96  98 100
---|----------------------------------------------------------------------------------------------------
 51| . 4 . . . 4 . 5 . . . 4 . 3 . . . . . 4 . . . 4 . 6 . . . 4 . 5 . . . 4 . 5 . . . 4 . 5 . . . 3 . 3
 52| . .(3). 3 . 3 . 6 . 4 . 4 . . . 5 . 4 . 5 . 4 . 5 . 4 . 3 . 4 . 4 . 3 . 4 . 4 . . . 4 . 5 . 3 . 3 .
 53| . . . 3 . 4 . 4 . 3 . 3 . 4 . 4 . 4 . 4 . 4 . 4 . 4 . 4 . 4 . 4 . 3 . 5 . 5 . 3 . 5 . 6 . 6 . 4 . 4
 54| . . . .(3). . . 3 . 4 . . . 4 . 3 . . . 4 . 5 . . . 4 . 3 . . . 4 . 3 . . . 3 . 3 . . . 3 . 4 . . .
 55| . . . . . 4 . 4 . . . 4 . 4 . . . 3 . . . 7 . 3 . 4 . 4 . . . 3 . 4 . 4 . . . . . . 3 . 4 . 5 . 5 . .
 56| . . . . . . 4 . 3 . 4 . . . 4 . 3 . 3 .(3). 4 . 5 . . . 3 . 5 . 4 . 4 . 5 . 4 . . . 4 . 3 . 4 . 4 .
 57| . . . . . . . 5 . . . 5 . 6 . . . 5 . 5 . . . 4 . . . . . 4 . 4 . . . 4 . 6 . . . 4 . 6 . . . 3 . 5
 58| . . . . . . . . 3 . 4 . 3 . 4 . 3 . 5 . 3 . 5 . 6 . 5 . 4 . 4 . 3 . 5 . . . 4 . 3 . 3 . 4 . 3 . 3 .
 59| . . . . . . . . . 4 . 4 . 4 . 4 . 5 . 4 . 3 . 5 . 5 . 5 . 4 . 5 . 5 . 4 . 3 . 4 . 5 . 4 . 6 . 4 . 4
 60| . . . . . . . . . . 4 . . . . . 3 . . . 3 . 5 . . . 4 . 3 . . .(3). . . . . 3 . 5 . . . . . 4 . . .
 61| . . . . . . . . . . . 4 . 5 . 3 . 3 . 4 . 4 . 5 . 3 . 3 . 4 . 5 . 5 . 5 . 3 . 4 . 3 . 3 . 4 . 3 . 5
 62| . . . . . . . . . . . . 4 . 5 . 5 . 5 . 3 . 4 . 3 . 4 . 7 . 4 . 4 . 4 . 4 . 3 . 4 . . . 5 . 4 . 4 .
 63| . . . . . . . . . . . . . 4 . . . 5 . . . . . 3 . 4 . . . 4 . 4 . . .(3). 5 . . . 5 . 4 . . . . . 3
 64| . . . . . . . . . . . . . . 3 . 3 . 4 . 4 . 4 . 4 . 4 . 3 . 6 . 5 . 4 . 4 . 3 . 4 .(3). 4 . 3 . 4 .
 65| . . . . . . . . . . . . . . . 5 . 4 . . . 4 . 3 . 5 . . . . . 3 . 4 . 5 . 4 . . . 4 . 3 . 3 . 4 . .
 66| . . . . . . . . . . . . . . . . 3 . . . 3 . 4 . . . . . 3 . . .(4). 4 . . . 4 . 5 . . . 6 . 4 . . .
 67| . . . . . . . . . . . . . . . . . 5 . 4 . 3 . 4 . 4 . 5 . 5 . 5 . 5 . 5 . 3 . 4 . 4 . 4 . 6 . 4 . 4
 68| . . . . . . . . . . . . . . . . . . 4 . 7 . 6 . 4 . 4 . 4 . 3 . 4 . . . 4 . 5 . 4 . 5 . 4 . 5 . 4 .
 69| . . . . . . . . . . . . . . . . . . . 3 . . . 4 . 3 . . . 3 . 3 . . . 4 . 6 . . . . . 4 . . . 3 . 5
 70| . . . . . . . . . . . . . . . . . . . . 5 . 5 . . . . . 4 . 3 . 4 . . . 4 . 4 . . . 3 . . . 4 . 3 .
 71| . . . . . . . . . . . . . . . . . . . . . 3 .(4). 6 . 5 . 5 . 4 . 4 . 5 . 3 . 4 . 3 . 4 . 4 . 5 . 5
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 81| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6 . . . 5 . 6 . . . 3 . 3 . . . 4 . 4
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 84| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 . . . 3 . . . . . 3 . 3 . . .
 85| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 . 5 . . . 5 . 4 . 3 . 5 . .
 86| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 . 4 . 3 . 3 . 3 . 3 . 5 .
 87| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5 . . . 3 . 4 . . . 5 . 5
 88| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3 . 4 . 3 . 4 . 4 . . .
 89| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .(3). 5 . 3 . 4 . 4 . 4
 90| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 . . . . . 5 . . .
 91| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 . 4 . 3 . . . 4
 92| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 . 3 . 4 . 4 .
 93| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5 . . . 3 . 3
 94| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 . 5 . 5 .
 95| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5 . 4 . .
 96| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 . . .
 97| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5 . 3
 98| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4 .
 99| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 4
100| . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
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