Heron triangles with three rational medians*

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Abstract

We study the eight elliptic curves whose rational points correspond to Heron triangles with two rational medians. We show that none of these triangles can have three rational medians.

Keywords : rational area triangle, elliptic curve, Chabauty method.

Introduction

In the attempt to find all triangles with three rational sides and rational area (commonly called Heron triangles) with the property that they also have three rational medians (see problem D21 of [7]) previous authors, [4, 5, 1], have uncovered eight infinite families of Heron triangles with two rational medians.

In [1] the authors show that these families correspond to eight elliptic curves, all isomorphic to each other. The subsequent exploration of these curves revealed that constraining the remaining median to be rational required one to find rational points on a genus seven curve—which by Faltings' theorem leads to a finite number of possible solutions. These were left unresolved.

In this paper we dispose of the unresolved finite list of solutions in the sense that we find them all and verify that none of them correspond to non-trivial Heron triangles with three rational medians.

We hasten to add that this does not solve D21 since there are an unknown number of Heron triangles with two rational medians which do not lie on these eight curves. The four known "sporadic" triangles (from [1]), have been checked—and found wanting.

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Defining equations

All rational sided triangles, (a, b, c) say, with two rational medians are parameterized by

$$
\begin{aligned}
a &= (-2\phi\theta^2 - \phi^2\theta) + (2\theta\phi - \phi^2) + \theta + 1 \\
b &= (\phi\theta^2 + 2\phi^2\theta) + (2\theta\phi - \theta^2) - \phi + 1 \\
c &= (\phi\theta^2 - \phi^2\theta) + (\theta^2 + 2\theta\phi + \phi^2) + \theta - \phi
\end{aligned}
\tag{1}
$$

(see [3]) where θ and ϕ are arbitrary rational parameters. Of the three medians, defined by the equations

$$
\begin{aligned}
4k^2 &= 2b^2 + 2c^2 - a^2 \\
4l^2 &= 2c^2 + 2a^2 - b^2 \\
4m^2 &= 2a^2 + 2b^2 - c^2,
\end{aligned}
\tag{2}
$$

we find that two, k and l, are automatically rational while the third, m, is not necessarily rational. Substituting the sides into the m-equation leads to a surface some of whose rational points correspond to triangles with a third rational median satisfying

$$
\begin{aligned}
S : 4m^2 = 4 &+ 9\phi^2\theta^4 - 4\phi + 18\theta\phi + 4\theta + 6\phi\theta^2 - 6\phi^2\theta - 6\phi\theta^4 - 22\theta\phi^3 \\
&+ 6\phi^2\theta^2 + 6\phi^4\theta + 9\phi^4\theta^2 - 22\phi\theta^3 + 18\phi^3\theta^2 - 18\phi^2\theta^3 \\
&+ 18\phi^3\theta^3 - 3\phi^2 - 3\theta^2 + \phi^4 + \theta^4 - 2\theta^3 + 2\phi^3 = f_4(\theta, \phi).
\end{aligned}
\tag{3}
$$

The area, Δ, can be expressed directly in terms of the sides of the triangle by using Heron's formula, $\Delta^2 = s(s-a)(s-b)(s-c)$ where s is the semi-perimeter. We prefer to write it in the form

$$
T : 16\Delta^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 \tag{4}
$$

or, when expressed in terms of θ and ϕ, as $T_{\theta\phi}$:

$$
\Delta^2 = 16\theta\phi(\theta^2 - 1)(\phi^2 - 1)(3\theta\phi + \theta - \phi + 1)(2\theta + \phi - 1)(\theta + 2\phi + 1)(\theta - \phi + 1). \tag{5}
$$

In [1] it was shown that any point (θ, ϕ) which lies on one of the eight curves defined by

$$
\begin{aligned}
&C_1 : 27\theta^3\phi^3 - \theta\phi(\theta - \phi)(8\theta^2 + 11\theta\phi + 8\phi^2) - 3\theta\phi(5\theta^2 - \theta\phi + 5\phi^2) \\
&\qquad - (\theta - \phi)(\theta^2 + 4\theta\phi + \phi^2) - 3\theta^2 + 7\theta\phi - 3\phi^2 - 3\theta + 3\phi - 1 \\
&C_2 : 3\theta^2\phi^2 - 2\theta\phi(\theta - \phi) - \theta^2 - 6\theta\phi - \phi^2 + 1 \\
&C_3 : \theta\phi(\theta - \phi)^3 - \theta^4 - 11\theta^3\phi - 3\theta^2\phi^2 - 11\theta\phi^3 - \phi^4 - 2\theta^3 + 2\phi^3 \\
&\qquad + 10\theta\phi + 2\theta - 2\phi + 1 \\
&C_4 : \theta\phi(\theta - \phi) + \theta\phi + 2(\theta - \phi) - 1 \\
&C_5 : (\theta - 1)^3\phi^2 + 2(\theta + 1)(\theta^3 + 2\theta^2 - 2\theta + 1)\phi + (2\theta - 1)(\theta + 1)^3 \\
&C_6 : \phi^4 + 2\theta\phi^3 - \phi^3 - 3\phi^2\theta - 3\phi\theta^2 - 2\theta\phi - \theta - \theta^2 \\
&C_7 : (-\phi - 1)^3\theta^2 + 2(\phi - 1)(-\phi^3 + 2\phi^2 + 2\phi + 1)\theta - (2\phi + 1)(1 - \phi)^3 \\
&C_8 : 3\phi^2\theta - \phi^2 + 3\phi\theta^2 - 2\theta\phi + 2\theta^3\phi + \phi + \theta^3 + \theta^4
\end{aligned}
\tag{6}
$$

has the property that the corresponding triangle with two rational medians also has rational area. Thus our study of rational triangles with three rational medians and rational area leads naturally to consideration of the sets

$$C_i(\mathbb{Q}) \cap S(\mathbb{Q})$$

for $i = 1, \ldots, 8$.

A simple reduction

Consider the mapping of the (θ, ϕ)-plane to itself defined by

$$\mu : (\theta, \phi) \longrightarrow (-\phi, -\theta)$$

which, when applied to the expressions for (a, b, c) in equations (1), leads to

$$(a, b, c)^\mu = (b, a, c).$$

In other words, this simply flips the triangle about the m-median leaving the area and the third median unchanged. So when μ is applied to the median surface (3) and Heron surface (5), we get

$$S^\mu = S \quad \text{and} \quad T^\mu = T$$

while when it is applied to the curves (6) one gets

$$C_i^\mu = C_i \text{ for } i = 1...4, \quad C_5^\mu = C_7, \text{ and } C_6^\mu = C_8.$$

Since μ is an isomorphism preserving all the rational points we see that it is sufficient to consider just the 6 sets

$$C_i(\mathbb{Q}) \cap S(\mathbb{Q})$$

for $i = 1, \ldots, 6$.

Rational points on one of the covers

In this section we compute the finite set $C_4(\mathbb{Q}) \cap S(\mathbb{Q})$ and then show in a corollary that none of these points correspond to Heron triangles with three rational medians. Since it was already shown, in [1], that a covering curve for this set has genus 7 we need a tool to reduce the complexity and make it more manageable.

If one defines the σ-transformation of the (θ, ϕ)-plane via

$$\sigma : (\theta\phi, \theta - \phi) \longrightarrow (\theta, \phi)$$

then one observes that it is a rational transformation which can be applied to the first four curves $C_1, \ldots, C_4$ and crucially, to S. This is one of the tools we use in the following main theorem.

Theorem 1 *The simultaneous rational solutions to C_4 and S are given by*

$$C_4(\mathbb{Q}) \cap S(\mathbb{Q}) = \left\{ \left(0, -\tfrac{1}{2}\right), \left(\tfrac{1}{2}, 0\right), (1,-1), (-1,3), (-3,1), (-1,-1), (1,1) \right\}.$$

Proof: Note that, since σ is a rational map, all the rational points on $C_4(\mathbb{Q}) \cap S(\mathbb{Q})$ map to rational points on $C_4^\sigma(\mathbb{Q}) \cap S^\sigma(\mathbb{Q})$. Thus if we can find all the rational points in the set $C_4^\sigma(\mathbb{Q}) \cap S^\sigma(\mathbb{Q})$ then their preimages will be all the rational points in the set $C_4(\mathbb{Q}) \cap S(\mathbb{Q})$.

First we compute

$$C_4^\sigma : \theta\phi + \theta + 2\phi - 1$$

and S^σ :

$$4m^2 = 36\theta^3 + 9\theta^2\phi^2 - 36\theta^2\phi - 36\theta^2 - 6\theta\phi^3 - 18\theta\phi^2 + 12\theta + \phi^4 - 2\phi^3 - 3\phi^2 + 4\phi + 4.$$

If we let D_4^σ denote the resultant of C_4^σ and S^σ, from which we eliminate ϕ, then the common rational solutions to C_4^σ and S^σ are also the common rational solutions to C_4^σ and D_4^σ. We ignore the conic C_4^σ since it has solutions and concentrate instead on D_4^σ. A short computation reveals that

$$D_4^\sigma : 4(\theta+2)^4 m^2 = 9(\theta+1)(\theta^2+4\theta+1)(4\theta^4+13\theta^3+\theta^2-15\theta+9)$$

is a genus 3 hyperelliptic curve. We use the bi-rational transformation

$$(x,y) = \left(\theta, \frac{2(\theta+2)^2 m}{3}\right)$$

to put it into the form

$$H : y^2 = (x+1)(x^2+4x+1)(4x^4+13x^3+x^2-15x+9).$$

Observe that the only Weierstrass point in $H(\mathbb{Q})$ is $(-1,0)$. To find the non-Weierstrass rational points of H we consider the covering curves, F_λ

$$F_\lambda \quad : \quad \begin{cases} \lambda y_1^2 & = (x^2+4x+1) = f_1(x) \\ \lambda y_2^2 & = (x+1)(4x^4+13x^3+x^2-15x+9) = f_2(x) \end{cases} \tag{7}$$

where λ is squarefree and divides $\gcd(f_1(x), f_2(x)) = 48$, *i.e.*, $\lambda \mid 6$. The eight cases to consider break up into two similar families.

- When $\lambda = -1, 2, 3, -6$ it is relatively easy to show that the conics are all empty. For example, when $\lambda = -1$ the conic part of F_{-1} is $y_1^2 + (x+2)^2 = 3$ which can be shown to have no solutions over $\mathbb{Q}$ by simply homogenising and showing there are no solutions modulo 3. The conics corresponding to the other three values of λ can be dealt with in a similar way.
- When $\lambda = 1, -2, -3, 6$ the partner curve to each conic can be shown to be a genus 2, rank 1, hyperelliptic curve. A significantly more sophisticated argument relying on Chabauty's Theorem can then be used to find all the rational points. When $\lambda = 1$ we have the curve

$$H_1 : y_2^2 = (x+1)(4x^4+13x^3+x^2-15x+9)$$

and a short search reveals the four points[1]

$$\{\infty, (-1,0), (0,-3), (0,3)\}$$

of height less than 50. While the rational points on H_1 do not form a group, the divisor classes of degree 0 defined over $\mathbb{Q}$ do form a finitely generated abelian group which we denote by $G(H_\lambda)$. We represent the divisors, as do Cassels and Flynn [6], as pairs of points. In particular, if we let

$$A = [\infty, (0,-3)] \quad \text{and} \quad B = [\infty, (-1,0)]$$

then A is an infinite order divisor while B is a divisor of order 2. In fact, a Selmer group computation in `Magma`[2] [2], to determine the number of infinite components, and a series of finite field computations, to determine the torsion subgroup shows that

$$G(H_1) \cong \mathbb{Z} \times \frac{\mathbb{Z}}{2\mathbb{Z}}.$$

A descent computation proves that the divisor A is in fact the generator of the infinite component—thus $G(H_1) = \langle A, B\rangle$. Finally, an application of Chabauty's theorem ([6]) using the infinite order generator A shows that all the rational points on H_1 are given by

$$H_1(\mathbb{Q}) = \{\infty, (-1,0), (0,\pm 3), (55/9, \pm 179232/9)\}.$$

Of these points, the only ones that also satisfy the conic in the pair F_1 and hence map to rational points on H are

$$\{(-1,0), (0,\pm 3)\}.$$

Recall that C_4^σ implies that $\phi = (1-\theta)/(2+\theta)$ so the only affine points on $C_4^\sigma \cap D_4^\sigma$ and hence on $C_4^\sigma \cap S^\sigma$ are

$$(\theta, \phi, m) = (0, 1/2, \pm 9/8), (-1, 2, 0).$$

The final mapping back to $C_4 \cap S$ is given by undoing σ, namely

$$\{(\theta,\phi) : \theta\phi = 0, \theta - \phi = 1/2\} \cup \{(\theta,\phi) : \theta\phi = -1, \theta - \phi = 2\}$$

hence $(\theta, \phi) = (0, 1/2), (-1/2, 0), (1, -1)$.

When $\lambda = -2$ we find that $G(H_{-2}) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and in fact

$$G(H_{-2}) = \langle A, B\rangle \text{ where } A = [\infty, (-3,-12)], B = [\infty, (-1,0)].$$

Chabauty's theorem gives us all the rational points on H_{-2}, namely,

$$H_{-2}(\mathbb{Q}) = \{\infty, (-1,0), (-3,\pm 6), (-41/32, \pm 5601/4096)\}.$$

These lead to two new points on H,

$$(x,y) = (-3, \pm 12) \in H(\mathbb{Q})$$

[1]Note that $\infty = [1:0:0]$ in homogeneous coordinates unless otherwise stated.
[2]Version 2.15-8

both of which correspond to $(\theta, \phi) = (-3, -4)$ on D_4^σ. Finding the preimages gives us $(\theta, \phi) = (-1, 3), (-3, 1)$ on $C_4 \cap S$.

When $\lambda = -3$ we find that $G(H_{-3}) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and in fact

$$G(H_{-3}) = \langle A, B \rangle \quad \text{where} \quad A = [\infty, (-2, -3)], B = [\infty, (-1, 0)].$$

Chabauty's theorem gives us all the rational points on H_{-3}, namely,

$$H_{-3}(\mathbb{Q}) = \{\infty, (-1, 0), (-2, \pm 1)\}.$$

These lead to two new points on H,

$$(x, y) = (-2, \pm 3) \in H(\mathbb{Q})$$

neither of which correspond to affine points on D_4^σ. In particular, we get no new points on $C_4 \cap S$.

When $\lambda = 6$ we find that $G(H_6) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and in fact

$$G(H_6) = \langle A, B \rangle \quad \text{where} \quad A = [\infty, (1, -12)], B = [\infty, (-1, 0)].$$

Chabauty's theorem gives us all the rational points on H_6, namely,

$$H_6(\mathbb{Q}) = \{\infty, (-1, 0), (1, \pm 2)\}.$$

These lead to two new points on H,

$$(x, y) = (1, \pm 12) \in H(\mathbb{Q})$$

both of which correspond to $(\theta, \phi) = (1, 0)$ on D_4^σ. Finding the preimages gives us $(\theta, \phi) = (-1, -1), (1, 1)$ on $C_4 \cap S$. □

Notice that in the process of proving the theorem we ended up determining all the rational points on the genus 3 hyperelliptic curve H, namely

$$H(\mathbb{Q}) = \{\infty, (0, \pm 3), (-1, 0), (1, \pm 12), (-2, \pm 3), (-3, \pm 12)\}.$$

where the point at infinity is $[0 : 1 : 0]$ in homogeneous coordinates.

Corollary 1 *No Heron triangle with 2 rational medians corresponding, via equations (1), to a rational point on C_4 has 3 rational medians.*

Proof: Simply mapping the 7 affine points of $C_4(\mathbb{Q}) \cap S(\mathbb{Q})$, from the previous theorem, via equations (1), gives us the collection of solutions to (2) and (4), namely,

$$\left(\tfrac{3}{4}, \tfrac{3}{2}, \tfrac{3}{4}\right), \left(\tfrac{3}{2}, \tfrac{3}{4}, \tfrac{3}{4}\right), (0, 0, 0), (-12, -24, 12), (-24, -12, 12), (4, 0, 4), (0, 4, 4)$$

none of which form proper triangles. □

The remaining covers

In this section we will show that the remaining five cases, namely $C_i : i \in \{1, 2, 3, 5, 6\}$, are reducible to just that involving C_4. First we recall a few transformations of the (θ, ϕ)-plane which preserve solutions to the k and l equations from (2). We define three maps

$$A : a \longrightarrow -a, \qquad C : c \longrightarrow -c, \qquad K : k \longrightarrow -k \tag{8}$$

where the action on the other variables, from (a, b, c, k, l), is trivial. The representation of these maps in the (θ, ϕ) plane is given by

$$\begin{aligned} (\theta,\phi)^A &= \left(\frac{\theta\phi + \theta - \phi^2 + 1}{\phi(2\theta + \phi - 1)}, \frac{3\theta\phi + \theta - \phi + 1}{2\theta^2 + \theta\phi + \theta + \phi - 1} \right) \\ (\theta,\phi)^C &= \left(\frac{\phi(2\theta + \phi - 1)}{\phi^2 - \theta\phi - \theta - 1}, \frac{\theta(2\phi + \theta + 1)}{\theta\phi - \theta^2 - \phi + 1} \right) \\ (\theta,\phi)^K &= \left(\theta, \frac{1 - \theta - \phi - \theta\phi - 2\theta^2}{3\theta\phi + \theta - \phi + 1} \right). \end{aligned} \tag{9}$$

One can confirm, as was done in [1], that

$$C_1^A = C_2^{AC} = C_3^C = C_5^K = C_6^{AK} = C_4.$$

While this implies that (all but a finite number of) the rational points on the 5 other curves can be mapped to the curve C_4 (and vice versa) we need to ensure that the transformations also preserve the rationality of the third median and the rationality of the area in the process. Notice that A, C, and K are involutions, while AC and AK have CA and KA as their inverse maps.

The first 2 transformations, A and C, almost trivially preserve $m \in \mathbb{Q}$ and $\Delta \in \mathbb{Q}$, since,

$$S^A = S^C = S \quad \text{and} \quad T^A = T^C = T$$

by applying (8) to equations (2) and (4). The compositions AC and its inverse CA also have the same property.

For the transformation K we need to work a little harder. When we apply K directly to the equation defining the surface $S : 4m^2 = f_4(\theta, \phi)$ we obtain

$$S^K : 4(m^K)^2 = \frac{2^2 3^2 \theta^2 (\theta - 1)^2 (\theta + 1)^2}{(3\theta\phi + \theta - \phi + 1)^4} f_4(\theta, \phi)$$

which implies that if m was originally rational then m^K is also rational.

Similarly, applying K directly to the area equation $T_{\theta\phi} : \Delta^2 = f_7(\theta, \phi)$ leads to

$$T_{\theta\phi}^K : (\Delta^K)^2 = \frac{3^4 \theta^4 (\theta - 1)^4 (\theta + 1)^4}{(3\theta\phi + \theta - \phi + 1)^8} f_7(\theta, \phi)$$

which as above, shows that the area of the transformed triangle is rational if and only if the original area was rational.

Now, to obtain all the rational points in the remaining 5 cases, we need to re-express the curves and the maps in homogeneous coordinates. If we let $\theta = X/Z$ and $\phi = Y/Z$ then the curve C_4 becomes

$$C_4 : XY(X-Y) + XYZ + 2(X-Y)Z^2 - Z^3 = 0.$$

There are 3 new rational points corresponding to $Z = 0$ so the homogeneous solutions to Theorem 1 are

$$\begin{aligned} C_4(\mathbb{Q}) \cap S(\mathbb{Q}) = \{&[0:-1:2],[-3:1:1],[1:0:2],[1:1:1],[1:-1:1], \\ &[-1:3:1],[-1:-1:1],[1:0:0],[0:1:0],[1:1:0]\}. \end{aligned}$$

The map A in homogeneous coordinates becomes

$$\begin{aligned} A([X:Y:Z]) = [&(X+Z)(Y+Z)(X-Y+Z) : \\ &(3XY + XZ - YZ + Z^2)Y : \\ &(2X^2 + XY + XZ + YZ - Z^2)Y]. \end{aligned}$$

The singular (or undefined) points of the map $A : P^2(\mathbb{Q}) \mapsto P^2(\mathbb{Q})$, denoted by $sing(A)$, occur when $A([X:Y:Z]) = [0:0:0]$ and are given by

$$sing(A) = \{[0:1:0],[1:0:0],[0:1:1],[1:-1:1],[-1:0:1]\}.$$

To find all the rational points on $C_i(\mathbb{Q}) \cap S(\mathbb{Q})$, say, we need to consider the union of 3 sets. Referring to the sequence

$$C_4 \xrightarrow{\alpha} \boxed{C_i} \xrightarrow{\beta} C_4 \xrightarrow{\alpha} C_i$$

these sets are:

- the mapped points $[X:Y:Z] \in \{C_4(\mathbb{Q}) \cap S(\mathbb{Q})\}^{\alpha}$,
- the singularities of the map $\beta : C_i \mapsto C_4$, and
- the β-preimages of the singularities of the map $\alpha : C_4 \mapsto C_i$.

Finally, we apply the above technique to $C_i(\mathbb{Q}) \cap S(\mathbb{Q})$ for each of the five remaining curves, namely, $i = 1, 2, 3, 5, 6$. For example, when $i = 1$ we get the mapped set:

$$\begin{aligned} \{C_4(\mathbb{Q}) \cap S(\mathbb{Q})\}^A = \{&[1:-1:1],[1:-1:1],[1:0:0],[1:1:1],\text{undef}, \\ &[0:1:0],[0:1:0],\text{undef},\text{undef},[0:1:1]\}, \end{aligned}$$

the singularities of A on C_1:

$$sing(A : C_1 \mapsto C_4) = \{[-1:0:1],[0:1:1],[1:-1:1],[0:1:0],[1:0:0]\},$$

and the preimages of the singularities of A on C_4:

$$A^{-1}(sing(A : C_4 \mapsto C_1)) = \{[-1:-1:1]\}.$$

So we have the complete set of rational points on $C_1 \cap S$, namely,

$$C_1(\mathbb{Q}) \cap S(\mathbb{Q}) = \{[1:-1:1],[1:0:0],[1:1:1],[0:1:0],[0:1:1], [-1:0:1],[-1:-1:1]\},$$

which correspond to solutions of equations (2) and (4), namely,

$$(0,0,0),(0,0,0),(0,4,4),(0,0,0),(0,0,0),(0,0,0),(4,0,4)$$

none of which form non-degenerate Heron triangles with three rational medians.

A similar computation for the remaining curves, C_2, C_3, C_5, C_6, provides the complete list of rational points on each cover in Table 1. The affine solutions in turn correspond to the

cover	#	solutions
$C_1(\mathbb{Q}) \cap S(\mathbb{Q})$	7	$[1:1:1],[0:1:0],[-1:0:1],[0:1:1],$ $[1:0:0],[1:-1:1],[-1:-1:1]$
$C_2(\mathbb{Q}) \cap S(\mathbb{Q})$	8	$[-2:-1:1],[1:0:1],[1:2:1],[1:0:0],$ $[0:-1:1],[0:1:1],[0:1:0],[-1:0:1]$
$C_3(\mathbb{Q}) \cap S(\mathbb{Q})$	8	$[1:0:1],[1:0:0],[1:-1:1],[0:-1:1],$ $[0:1:1],[0:1:0],[1:1:0],[-1:0:1]$
$C_4(\mathbb{Q}) \cap S(\mathbb{Q})$	10	$[0:-1:2],[1:0:2],[1:-1:1],[-1:3:1],$ $[-3:1:1],[-1:-1:1],[1:1:1],[1:0:0],$ $[0:1:0],[1:1:0]$
$C_5(\mathbb{Q}) \cap S(\mathbb{Q})$	8	$[1/2:0:1],[1:0:0],[1:-1:1],[-3:1:1],$ $[0:1:1],[0:1:0],[1:1:1],[-1:0:1]$
$C_6(\mathbb{Q}) \cap S(\mathbb{Q})$	11	$[0:-1/2:1],[-1:-1:1],[1:0:0],[1:1:0],$ $[0:1:0],[-3:1:1],[1:-1:1],[1:1:1],$ $[-1:3:1],[1/2:0:1],[0:0:1]$

Table 1: Rational points on the six covers

distinct degenerate triangles shown in Table 2.

cover	#	triangles
$C_1(\mathbb{Q}) \cap S(\mathbb{Q})$	3	$(0,0,0),(0,4,4),(4,0,4)$
$C_2(\mathbb{Q}) \cap S(\mathbb{Q})$	5	$(12,-6,6),(2,0,2),(-6,12,6),(0,2,2),(0,0,0)$
$C_3(\mathbb{Q}) \cap S(\mathbb{Q})$	4	$(2,0,2),(0,0,0),(0,2,2),(0,0,0)$
$C_4(\mathbb{Q}) \cap S(\mathbb{Q})$	7	$\left(\frac{3}{4},\frac{3}{2},\frac{3}{4}\right),\left(\frac{3}{2},\frac{3}{4},\frac{3}{4}\right),(0,0,0),$ $(-12,-24,12),(-24,-12,12),(4,0,4),(0,4,4)$
$C_5(\mathbb{Q}) \cap S(\mathbb{Q})$	4	$\left(\frac{3}{2},\frac{3}{4},\frac{3}{4}\right),(0,0,0),(-24,12,12),(0,4,4)$
$C_6(\mathbb{Q}) \cap S(\mathbb{Q})$	8	$\left(\frac{3}{4},\frac{3}{2},\frac{3}{4}\right),(4,0,4),(-24,-12,12),(0,0,0),$ $(0,4,4),(-12,-24,12),\left(\frac{3}{2},\frac{3}{4},\frac{3}{4}\right),(1,1,0)$

Table 2: Degenerate Heron triangles with three rational medians

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