Cyclic Polygons with Rational Sides and Area*

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Abstract

We generalise the notion of Heron triangles to rational-sided, cyclic n-gons with rational area using Brahmagupta's formula for the area of a cyclic quadrilateral and Robbins' formulæ for the area of cyclic pentagons and hexagons. We use approximate techniques to explore rational area n-gons for n greater than six. Finally, we produce a method of generating non-Eulerian rational area cyclic n-gons for even n and conjecturally classify all rational area cyclic n-gons.

Keywords : Heron triangle, rational polygon, cyclic polygon, rational area.

1 Introductory Points

Around 200 BC, Archimedes, in a blatant act of anticipatory plagiarism, discovered Hero's formula for the area of a triangle in terms of the lengths of the sides ([21, p. 228], see also section 3). A short while later, in 628 AD, Brahmagupta concocted his less well known formula for the area of a cyclic quadrilateral ([6, pp. 56-59], see also section 4). Then, over 13 centuries later, Robbins presented the mathematics community with his derivation of the analogous formulæ for the areas of both cyclic pentagons and cyclic hexagons ([14, 15] see also sections 5 and 6).

Motivated by this flurry of activity we considered trying to find cyclic n-gons having rational sides and rational area. In the literature the well-studied case of $n = 3$ is usually called a Heron triangle. In this spirit we denote the case $n = 4$ by Brahmagupta quadrilateral, and the cases $n = 5, 6$ by Robbins pentagon and Robbins hexagon respectively.

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2 Notational Lines

We will use n-tuples enclosed in square brackets to denote the sides of our polygons, such as the triangle $[13, 14, 15]$, or sometimes in homogeneous form, like the square $[1 : 1 : 1 : 1]$. The area of the n-gon with sides $[a_1, \ldots, a_n]$ will be denoted by $K_n(a_1, \ldots, a_n)$ while we will use $R_n(a_1, \ldots, a_n)$ for the circumradius.

We also occasionally use a slightly less precise notation for the area of an n-gon in terms of its vertices. If the vertices are labelled (with uppercase letters) as $A_1, A_2, \ldots, A_n$ then the area of the convex hull of this collection is denoted by $K_n(A_1A_2 \ldots A_n)$ where the intent is that this is simply shorthand for $K_n(\overline{A_1A_2}, \overline{A_2A_3}, \ldots, \overline{A_{n-1}A_n})$.

We may drop either the subscripts or the functional dependence on the sides (or both) if it is obvious from context.

3 Heron Triangles

Recall that Hero's formula for the area, K_3 say, of a triangle in terms of the sides, a, b and c, is simply

$$K_3 = \sqrt{s(s-a)(s-b)(s-c)} \tag{1}$$

where $s = \frac{a+b+c}{2}$ is the semi-perimeter. Hero noted isolated examples of rational sided triangles with rational area like that with sides $[a, b, c] = [13, 14, 15]$ and area 84 (see [8, p. 191]).

One of the earliest examples of an infinite family of Heron triangles was given by Brahmagupta (see [8, p. 191]) sometime early in the 7th century AD. He observed that the triangle with sides given by

$$[a, b, c] = \left[\frac{u^2+v^2}{2v}, \frac{u^2+w^2}{2w}, \frac{u^2-v^2}{2v} + \frac{u^2-w^2}{2w}\right]$$

has rational area for any rational choice of u, v, w.

In 1621 Bachet described a method (see [8, p. 191-192]) of generating rational solutions to equation (1) by joining together two appropriately rescaled Pythagorean triangles (*ie.* right triangles with rational sides and area). Although this does provide a method of generating all Heron triangles, it was Euler ([8, p. 193]) who was the first to provably parametrize all such triangles via

$$[a : b : c] = [(ps+qr)(pr-qs) : rs(p^2+q^2) : pq(r^2+s^2)] \tag{2}$$

where p, q, r, s are arbitrary rational parameters. We describe Euler's method, of combining isosceles triangles, with repeated sides equal to the circumradius, in the next section. More recently, Carmichael provided a more economical parametrization ([3, pp. 12]) of all Heron triangles, namely

$$[a:b:c] = [n(m^2+k^2) : m(n^2+k^2) : (m+n)(mn-k^2)], \tag{3}$$

requiring only three rational parameters as opposed to Euler's four. It is easy to show that this is in fact equivalent to Brahmagupta's parametrization—so Carmichael's proof vindicates the primacy of the Indian mathematician.

Whenever we make a computational search for various cyclic n-gons with rational area it is expedient to seek out ways to reduce the number of cases to consider. One observation, for an integer sided Heron triangle, is that the perimeter is always even. Despite the fact that this is trivial to prove using either Euler's or Carmichael's parametrization, we give a proof, relying only on the area formula, which will be a guide in those cases when we do not have a parametrization to fall back on.

Theorem 1 *Any Heron triangle with three integer sides has integer area and even perimeter.*

Proof: Since $a, b, c \in \mathbb{Z}$ and the area is rational, then

$$(4K_3)^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4$$

immediately implies that $4K_3 \in \mathbb{Z}$. Furthermore, if an odd number of sides are odd then reducing the above equation modulo 4 leads to a contradiction, namely 3 is not a quadratic residue. Hence an even number of sides are odd and so K_3 is an integer, as is the semiperimeter $s = (a+b+c)/2$. □

At this stage we find it useful to define the notion of "radial decomposability" since it provides us with one useful means of determining if an n-gon in general is constructible from smaller rational area m-gons or not.

Definition: A cyclic n-gon with rational sides and area is *radially decomposable* if it can be subdivided into n isosceles Heron triangles each composed of two circumradii and one side of the n-gon.

Of course we are secretly interested in the indecomposable cyclic n-gons since they are the fundamental building blocks of all such n-gons. We should alert the reader that there are a number of special cases *e.g.* for right-angled triangles the decomposition is only into two proper isosceles triangles while for obtuse-angled triangles the sign of the triangle entirely outside must be made negative.

The following well known result is useful in the sequel.

Theorem 2 *Any Heron triangle is radially decomposable.*

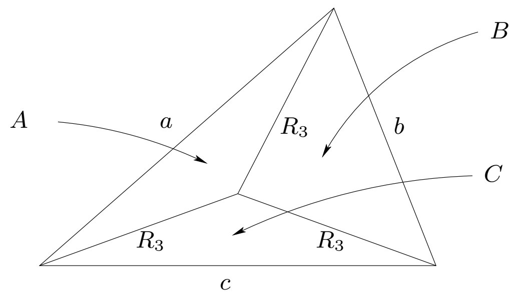

Figure 1: Radial decomposition of a Heron triangle

Proof: The circumradius, R_3, of a triangle, as that in Figure 1, with sides a, b and c and area K_3 is given (see [9]) by

$$R_3 = \frac{abc}{4K_3(a,b,c)}.$$

Clearly, for a rational-sided triangle the circumradius is rational if and only if the area is rational. Now let α, β, and γ be the vertex angles opposite sides a, b, and c, respectively. Since $K_3 = (bc\sin\alpha)/2$ and $\cos\alpha = (b^2 + c^2 - a^2)/2bc$, one observes that both $\sin\alpha$ and $\cos\alpha$ are rational. Thus, the isosceles wedge area $A = (R_3^2 \sin 2\alpha)/2 = R_3^2 \sin\alpha\cos\alpha$ is rational, as are the other two wedges, B and C. □

4 Brahmagupta Quadrilaterals

Brahmagupta's formula for the area, K_4 say, of a cyclic quadrilateral in terms of the sides, a, b, c and d, is simply

$$K_4 = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where $s = \frac{a+b+c+d}{2}$ again denotes the semi-perimeter. Since we will be considering a radial decomposition we need the circumradius formula

$$R_4 = \frac{\sqrt{(ac+bd)(ad+bc)(ab+cd)}}{4K_4} \tag{4}$$

first derived by Parameśvara around 1430 AD (see [10]).

Observe that if one sets one side, d say, to zero then the quadrilateral becomes a triangle and the area and circumradius formulæ above simply reduce to Heron's formula and the triangular circumradius formula for R_3. Unlike the Heron triangle case, the following theorem which restricts the perimeter actually forms a useful independent result.

Theorem 3 *Any Brahmagupta quadrilateral with four integer sides has integer area and even perimeter.*

Proof: Since the sides are integral and the area is rational, then

$$(4K_4)^2 = 8abcd + 2(a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) - a^4 - b^4 - c^4 - d^4$$

implies that $4K_4 \in \mathbb{Z}$. Just as in theorem 1 an odd number of sides cannot be odd as 3 is not a quadratic residue modulo 4. Thus $s = (a + b + c + d)/2$ is integral as is the area. □

For n-gons with 4 or more sides another notion of decomposability becomes possible—namely that of diagonal decomposability.

Definition: A cyclic n-gon with rational sides and area is *diagonally decomposable* if it can be subdivided into at least 2 disjoint rational area polygons using a common rational diagonal.

Now, initially we had not imagined that there was any relationship between diagonal decomposability and radial decomposability. However, a series of experiments suggested we should reverse our view. When we subsequently proved the appropriate theorems we had to admit that they were in fact the same concept, at least for quadrilaterals. The same appears to be true in the 5-gon case, at which point we were on the verge of believing it might be generally true. But a surprise was in store for us when we examined cyclic hexagons. Sadly, this surprise will have to wait while we first build up to our 4-gon conclusion.

Unlike Heron triangles there are two distinct types of Brahmagupta quadrilaterals, decomposable ones like the 3×4 rectangle and indecomposable ones like the unit square. We consider them each in turn.

4.1 Decomposable Quadrilaterals

First we note that Euler produced a parametrization of all rational-sided, radially decomposable cyclic n-gons (see [8, p. 221]). In 1905 Schubert ([18, pp. 28-38]) produced a parametrization of diagonally decomposable Brahmagupta quadrilaterals, while a more recent version was given by Sastry ([16, 17]). When n is 4, these all turn out to be equivalent parametrizations.

We specialise Euler's method to cyclic quadrilaterals. Since the circumradius is rational it is sufficient to generate all such 4-gons on the unit circle and then scale them up by an arbitrary rational parameter. Euler chooses 3 rational parameters p_1, p_2, p_3 which are used to generate 3 angles, labelled $\theta_1, \theta_2, \theta_3$, via the usual half-angle formulæ

$$\sin\theta_i = \frac{2p_i}{p_i^2+1}, \qquad \text{and} \qquad \cos\theta_i = \frac{p_i^2-1}{p_i^2+1}.$$

If we want a convex 4-gon then these three angles must satify

$$0 < \theta_i \le \frac{\pi}{2}, \qquad \text{and} \qquad \sum_{i=1}^{3} \theta_i < \pi.$$

Euler also defines the remaining angle, θ_4, by

$$\theta_4 = \pi - \sum_{i=1}^{3} \theta_i.$$

The θ_i are in fact the half-angles of the angle subtended by each side of a corresponding 4-gon, at the centre of the circumcircle (see Figure 2). Notice

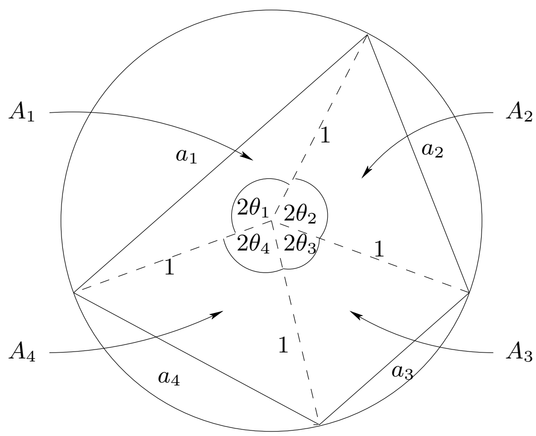

Figure 2: A radially decomposable cyclic quadrilateral

that the circumcentre lies inside the 4-gon, outside the 4-gon or on the side a_4 when $\theta_4 < \pi/2$, $\theta_4 > \pi/2$ or $\theta_4 = \pi/2$, respectively. Now since $\sin(A \pm B)$ and $\cos(A \pm B)$ are rationally expressible in terms of $\sin A$, $\sin B$, $\cos A$, and $\cos B$ Euler finds that both $\sin\theta_4$ and $\cos\theta_4$ are rational. Hence he can conclude that all the side lengths and wedge areas of the corresponding 4-gon, given by

$$a_i = \sqrt{2 - 2\cos 2\theta_i} = 2\sin\theta_i$$
$$A_i = \frac{1}{2}\sin 2\theta_i = \sin\theta_i \cos\theta_i$$

are rational. Thus the 4-gon has rational area and rational sides.

Next we study the relationship between radially decomposable quadrilaterals and diagonally decomposable quadrilaterals. Recall that Ptolemy's first quadrilateral theorem (see [9]), which when applied to Figure 3, states that $u_1 u_2 = ac + bd$ for any cyclic quadrilateral. This immediately implies that one diagonal is rational precisely when the other diagonal is rational. The following lemma provides the functional dependence of the diagonals on the sides.

Lemma 1 *In any cyclic quadrilateral with sides a, b, c, d the diagonals are given by*

$$u_1 = \sqrt{\frac{(ac+bd)(ad+bc)}{(ab+cd)}}, \qquad u_2 = \sqrt{\frac{(ac+bd)(ab+cd)}{(ad+bc)}}. \tag{5}$$

Proof: Let a, b, c, d denote the sides of the cyclic quadrilateral and u_1, u_2 the two diagonals as shown in Figure 3. Ptolemy's second quadrilateral theorem [9]

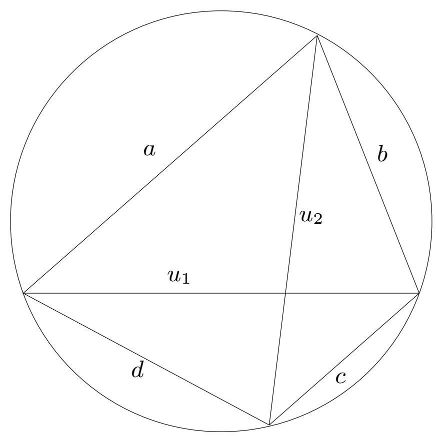

Figure 3: One rational diagonal implies two rational diagonals

is

$$\frac{u_1}{u_2} = \frac{ad+bc}{ab+cd}$$

which when multiplied by the formula for the product of the diagonals leads to

$$u_1^2 = \frac{(ac+bd)(ad+bc)}{(ab+cd)}.$$

Now just permute the sides to obtain the result for u_2. □

If we consider the relationship between the two types of diagonal decomposition, we readily obtain

Lemma 2 *Any Brahmagupta quadrilateral with one rational diagonal is diagonally decomposable along either diagonal.*

Proof: Let u_1 be the rational diagonal. If α is the angle between sides a and b and β is the angle between sides c and d then $\alpha + \beta = \pi$. In particular, notice that $\sin\alpha = \sin\beta$. Now the area of the triangles $K_3(a, b, u_1) = (ab\sin\alpha)/2$ while $K_3(c, d, u_1) = (cd\sin\beta)/2$ so that $K_4 = ((ab+cd)\sin\alpha)/2$. Since we are assuming that K_4 is rational then so are $\sin\alpha$, $\sin\beta$ and the areas $K_3(a, b, u_1)$ and $K_3(c, d, u_1)$. For the other decomposition apply Lemma 1. □

The first half of our claim connecting diagonal and radial decomposability is easy.

Theorem 4 *Any diagonally decomposable quadrilateral is radially decomposable.*

Proof: In the diagonally decomposable cyclic quadrilateral $ABCD$ (of Figure 4) the four triangles ABD, BCD, ACD, ABC are Heron triangles. By Theorem 2

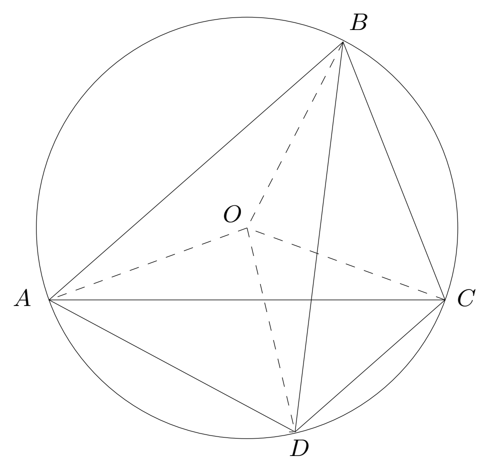

Figure 4: Diagonally decomposable quadrilateral

they are each radially decomposable and so each of the radial wedges AOB, BOC, COD, and DOA are Heron. The other three cases for which the circumcentre is on a diagonal, on a side, or outside the 4-gon are similar. □

To obtain the converse we require the following lemma.[1]

Lemma 3 *(Hughes) If two isosceles Heron triangles are joined along their common repeated sides, in such a way that the remaining repeated sides are adjacent, then the extra diagonal, created by joining the extreme vertices, is rational. Referring to Figure 5 if* $a, b, R, A, B \in \mathbb{Q}$ *then* $e \in \mathbb{Q}$.

Proof: Let α be the angle opposite side a and β be the angle opposite side b. Since these two isosceles triangles are Heron they have rational altitudes, and so one obtains

$$\sin\frac{\alpha}{2} = \frac{a}{2R}, \quad \sin\frac{\beta}{2} = \frac{b}{2R}, \quad \cos\frac{\alpha}{2} = \frac{2A}{aR}, \quad \cos\frac{\beta}{2} = \frac{2B}{bR}.$$

[1]The original proof was supplied to us by a colleague (Garry Hughes) who based the result on a formula relating the area of an arbitrary quadrilateral to its sides and diagonals. This formula has been attributed to various authors including Bretschneider (see [2]) and Coolidge (see [5]).

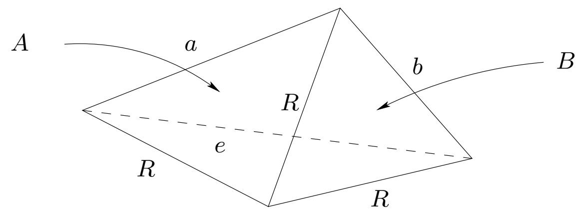

Figure 5: Joined isosceles triangles

Since the side we are interested in is given by

$$e = 2R\sin\left(\frac{\alpha+\beta}{2}\right) = 2R\left(\sin\frac{\alpha}{2}\cos\frac{\beta}{2} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\right)$$

it is clear that e is rational. □

Clearly, Lemma 3 immediately implies that any radially decomposable quadrilateral must have at least one rational diagonal. Combining this with Lemma 2 leads to a proof of the following:

Theorem 5 *Radially decomposable quadrilaterals are diagonally decomposable.*

So, the Eulerian quadrilaterals (mentioned earlier) completely describe all the radially (and diagonally) decomposable quadrilaterals. In an attempt to gain an understanding of all Brahmagupta quadrilaterals we discuss a few restricted families.

4.2 Restricted Families of Brahmagupta Quadrilaterals

By applying various restrictions to the sides of a quadrilateral we are led to simpler cases for which the corresponding rationality questions are more amenable to solution. For example, if the sides of a cyclic quadrilateral are in an arithmetic progression or a geometric progression then the area can never be rational (see [1]).

In the case of two equal sides it turns out to be fairly trivial to completely parametrize such isosceles Brahmagupta quadrilaterals. Suppose that the sides satisfy $[a, b, c, d] = [a, b, c, c]$, then the area is given by

$$K_4^2(a, b, c, c) = (a+b)^2(2c-a+b)(2c+a-b)$$

or $k^2 = 4c^2 - (a-b)^2$ where $k := K_4/(a+b)$. Now this last equation is a homogeneous quadratic and as such is amenable to the so-called chord method. All rational points on this (and in fact any) homogeneous quadratic can be parametrized by intersecting arbitrary slope chords through a fixed rational point on the curve, with the curve itself. If we dehomogenize by setting $(A, B, C) := (a/k, b/k, c/k)$ then we get the quadratic surface $4C^2 - (A - B)^2 = 1$ with particular solution $(A, B, C) = (0, 0, 1/2)$. We substitute

$$(A, B, C) = (0, 0, 1/2) + \lambda(p, q, r)$$

into the affine surface to obtain λ as a rational function of p, q, r. This then leads directly to the integer parametrization

$$\begin{aligned} ga &= 8pr \\ gb &= 8pq \\ gc &= 4p^2 + q^2 - 2qr + r^2 \\ gk &= 4p^2 - q^2 + 2qr - r^2 \end{aligned} \tag{6}$$

where g is simply the greatest common divisor of the 4 right hand sides and p, q, r are arbitrary integer parameters. This family will turn out to be quite useful since we will use it to generate radially indecomposable n-gons for $n > 5$.

As a second example we consider another three parameter family constrained so that the sides are of the form

$$[a, b, c, d] = [x - m, x + m, x - n, x + n]$$

for rational x, m, n. In this case, the semi-perimeter is $s = 2x$ so that the area is simply given by

$$K_4^2(a, b, c, d) = (x^2 - m^2)(x^2 - n^2).$$

Without loss of generality there exist rational parameters λ, α and β so that

$$\begin{aligned} x^2 - m^2 &= \lambda\alpha^2 \\ x^2 - n^2 &= \lambda\beta^2 \end{aligned}$$

making the area, $K_4 = \lambda\alpha\beta$, automatically rational. The two equations $x^2 = m^2 + \lambda\alpha^2$ and $x^2 = n^2 + \lambda\beta^2$ have general solutions

$$\begin{aligned} [m : \alpha : x] &= [r^2 - \lambda s^2 : 2rs : r^2 + \lambda s^2] \\ [n : \beta : x] &= [p^2 - \lambda q^2 : 2pq : p^2 + \lambda q^2] \end{aligned}$$

where p, q, r, s are rational. In the case where the greatest common divisors of the two right hand sides are one we can equate the two expressions for x to get

$$r^2 + \lambda s^2 = p^2 + \lambda q^2.$$

The chord method applied to this homogeneous quadratic gives the solution

$$[r : s : p : q] = [u^2 - \lambda v^2 + \lambda w^2 : 2uv : u^2 + \lambda v^2 - \lambda w^2 : 2uw].$$

Thus we get the family

$$\begin{aligned} x &= u^4 + 2\lambda u^2 v^2 + 2\lambda u^2 w^2 + \lambda^2 v^4 - 2\lambda^2 v^2 w^2 + \lambda^2 w^4 \\ m &= u^4 - 6\lambda u^2 v^2 + 2\lambda u^2 w^2 + \lambda^2 v^4 - 2\lambda^2 v^2 w^2 + \lambda^2 w^4 \\ n &= u^4 + 2\lambda u^2 v^2 - 6\lambda u^2 w^2 + \lambda^2 v^4 - 2\lambda^2 v^2 w^2 + \lambda^2 w^4. \end{aligned}$$

4.3 Indecomposable Quadrilaterals

What can we say about indecomposable rational area cyclic quadrilaterals, like the unit square, which have neither rational circumradii nor rational diagonals?

First we note that there are infinitely many such indecomposable Brahmagupta quadrilaterals by simply considering rectangles with side lengths $2u$ and $2v$ for integers u and v such that $0 < u < \sqrt{2v+1}$. The circumradius, R, given by $R = \sqrt{u^2+v^2}$, satisfies the inequality $v < R < v+1$—hence cannot be rational.

Secondly, despite the fact that these quadrilaterals are indecomposable, either diagonal divides them into two rational area triangles (thus proving the converse to Lemma 2 is false). If we denote one diagonal by u and the two areas either side by A and B then Ptolemy's theorem tells us that $u^2 \in \mathbb{Q}$ and Heron's formula shows that $A^2, B^2 \in \mathbb{Q}$. Then $A, B \in \mathbb{Q}$ follows from the identities

$$A = \frac{(A+B)^2 + A^2 - B^2}{2(A+B)} \quad \text{and} \quad B = \frac{(A+B)^2 + B^2 - A^2}{2(A+B)}.$$

Finally, by combining Parameśvara's formula (4) and Brahmagupta's formula, it is clear that the square of the circumradius is rational for any rational-sided cyclic quadrilateral, whether or not it has rational area. Thus we conclude that

$$R_4(a,b,c,d) \in \sqrt{m}\mathbb{Q}$$

for some squarefree positive integer m. When we focus on distinct similarity classes of quadrilaterals, we will show that R_4 can only be one of

$$1, \sqrt{2}, \sqrt{5}, \sqrt{10}, \sqrt{13}, \sqrt{17}, \ldots, \sqrt{m}, \ldots$$

where $m = u^2 + v^2$ for integers u and v (see Table 1).

a	b	c	d	area	radius	a	b	c	d	area	radius
1	1	1	1	1	$1/2\sqrt{2}$	6	6	5	5	30	$1/2\sqrt{61}$
2	2	1	1	2	$1/2\sqrt{5}$	7	5	5	1	16	$5/2\sqrt{2}$
3	3	1	1	3	$1/2\sqrt{10}$	7	7	1	1	7	$5/2\sqrt{2}$
3	3	2	2	6	$1/2\sqrt{13}$	7	7	2	2	14	$1/2\sqrt{53}$
4	4	1	1	4	$1/2\sqrt{17}$	7	7	3	3	21	$1/2\sqrt{58}$
4	4	3	3	12	$5/2$	7	7	4	4	28	$1/2\sqrt{65}$
5	5	1	1	5	$1/2\sqrt{26}$	7	7	5	5	35	$1/2\sqrt{74}$
5	5	2	2	10	$1/2\sqrt{29}$	7	7	6	6	42	$1/2\sqrt{85}$
5	5	3	3	15	$1/2\sqrt{34}$	8	5	5	2	20	$5/8\sqrt{41}$
5	5	4	4	20	$1/2\sqrt{41}$	8	6	3	1	12	$1/8\sqrt{1105}$
6	6	1	1	6	$1/2\sqrt{37}$						

Table 1: Brahmagupta quadrilaterals

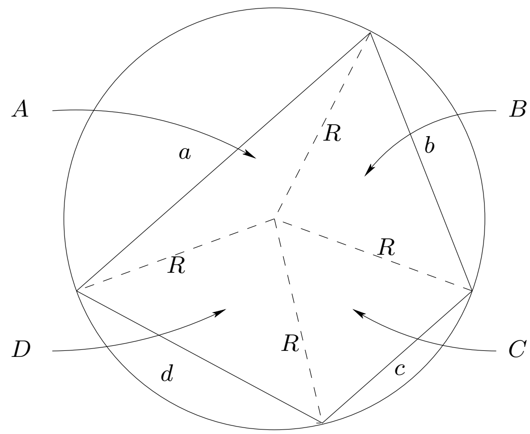

Figure 6: A cyclic quadrilateral

Now, when we apply Hero's formula to the isosceles triangle, $[a, R, R]$, in Figure 6 we get

$$\begin{aligned} A &= \sqrt{s(s-a)(s-R)(s-R)} \\ &= (s-R)\sqrt{s(s-a)} \\ &= \frac{a}{4}\sqrt{4R^2 - a^2} \end{aligned}$$

where $s = R + a/2$. Similarly, for the other three triangles giving us

$$A = \frac{a}{4}\sqrt{4R^2 - a^2}, \qquad B = \frac{b}{4}\sqrt{4R^2 - b^2}$$
$$C = \frac{c}{4}\sqrt{4R^2 - c^2}, \qquad D = \frac{d}{4}\sqrt{4R^2 - d^2}.$$

Since $R^2 \in \mathbb{Q}$ we infer that A^2, B^2, C^2, D^2 and $A + B + C + D$ are rational. A remarkable consequence (based on a paper by Ursell [20]) is that $A, B, C, D \in \mathbb{Q}$, hence all the radial triangles have rational area.[2] Without loss of generality we set $R := \sqrt{m}$ say, for some squarefree integer m, then since R must satisfy equations of the form

$$4R^2 - \alpha^2 = \beta^2$$

for rational α and β we see that m satisfies

$$m = \alpha^2 + \beta^2.$$

Characterising such integers is a well studied problem and the upshot is that

[2] This can be made quite explicit by setting $A + B + C + D = \alpha$, expressing the first four odd powers of α as linear equations in A, B, C, D with rational coefficients, and then solving this linear system.

the subset of squarefree ones correspond precisely to those of the form

$$m = \prod_{i=1}^{r} p_i$$

where p_i are primes such that $4 \nmid p_i + 1$. This proves our earlier comment on the sequence of possible circumradii of indecomposable quadrilaterals. Notice that for every m expressible as a sum of two integer squares, $u^2 + v^2$ say, we can always find a Brahmagupta quadrilateral with a circumradius of $\sqrt{m}$ by simply considering the rectangle with sides $[2u, 2v, 2u, 2v]$.

Can we determine all Brahmagupta quadrilaterals of a given circumradius?

Of course, we are really interested in the case when $R = \sqrt{m}$ for non-trivial squarefree m. It turns out that judicious use of the following lemma allows us to generalise Euler's method to precisely this setting.

Lemma 4 *If $m = u_0^2 + v_0^2$ then the general solution to*

$$m = u^2 + v^2$$

is given by

$$(u, v) = \left(\pm \frac{u_0 p^2 + 2v_0 p - u_0}{p^2 + 1}, \pm \frac{v_0 p^2 - 2u_0 p - v_0}{p^2 + 1} \right)$$

where $p \in \mathbb{Q}$.

Proof: Using the chord method we substitute $(u, v) = (u_0, v_0) + \lambda(P, Q)$ into the conic to obtain $\lambda = \frac{-2u_0 P - 2v_0 Q}{P^2 + Q^2}$. Thus we get rational functions for u and v in terms of P and Q

$$(u, v) = \left(\frac{-u_0 P^2 - 2v_0 PQ + u_0 Q^2}{P^2 + Q^2}, \frac{v_0 P^2 - 2u_0 PQ - v_0 Q^2}{P^2 + Q^2} \right)$$

which have the property that they have homogeneous quadratics as numerator and denominator. If we observe that the case $Q = 0$ corresponds to the solution $(-u_0, v_0)$ then without loss of generality we can set $p := P/Q$ to obtain our result. □

Thus, to construct all Brahmagupta quadrilaterals with radius $\sqrt{m}$ we choose 3 free rational parameters, p_1, p_2 and p_3 say, which correspond, via Lemma 4, to three angles, θ_1, θ_2 and θ_3 given by:

$$\sin \theta_i = \frac{u_0 p_i^2 + 2v_0 p_i - u_0}{\sqrt{m}(p_i^2 + 1)}$$

$$\cos \theta_i = \frac{v_0 p_i^2 - 2u_0 p_i - v_0}{\sqrt{m}(p_i^2 + 1)}$$

where u_0 and v_0 are integers satisfying $m = u_0^2 + v_0^2$. These three angles correspond to three sides of a quadrilateral since

$$\begin{aligned} a = 2\sqrt{m}\sin\theta_1 &= \frac{2(u_0p_1^2 + 2v_0p_1 - u_0)}{p_1^2+1} \\ b = 2\sqrt{m}\sin\theta_2 &= \frac{2(u_0p_2^2 + 2v_0p_2 - u_0)}{p_2^2+1} \\ c = 2\sqrt{m}\sin\theta_3 &= \frac{2(u_0p_3^2 + 2v_0p_3 - u_0)}{p_3^2+1}. \end{aligned}$$

If, as before, we define the remaining half-angle, θ_4 via $\theta_4 = \pi - \sum_{i=1}^{3}\theta_i$ then the same trigonometric identity, as used by Euler, namely

$$\begin{aligned} \sin\theta_4 =& \sin\theta_1\cos\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_2\cos\theta_3 + \cos\theta_1\cos\theta_2\sin\theta_3 \\ &- \sin\theta_1\sin\theta_2\sin\theta_3 \end{aligned}$$

shows that the remaining side, $d = 2\sqrt{m}\sin\theta_4$, is also rational

$$\begin{aligned} \frac{d}{2\sqrt{m}} = &\left(\frac{u_0p_1^2 + 2v_0p_1 - u_0}{p_1^2+1}\right)\left(\frac{v_0p_2^2 - 2u_0p_2 - v_0}{p_2^2+1}\right)\left(\frac{v_0p_3^2 - 2u_0p_3 - v_0}{p_3^2+1}\right) \\ &+ \left(\frac{v_0p_1^2 - 2u_0p_1 - v_0}{p_1^2+1}\right)\left(\frac{u_0p_2^2 + 2v_0p_2 - u_0}{p_2^2+1}\right)\left(\frac{v_0p_3^2 - 2u_0p_3 - v_0}{p_3^2+1}\right) \\ &+ \left(\frac{v_0p_1^2 - 2u_0p_1 - v_0}{p_1^2+1}\right)\left(\frac{v_0p_2^2 - 2u_0p_2 - v_0}{p_2^2+1}\right)\left(\frac{u_0p_3^2 + 2v_0p_3 - u_0}{p_3^2+1}\right) \\ &- \left(\frac{u_0p_1^2 + 2v_0p_1 - u_0}{p_1^2+1}\right)\left(\frac{u_0p_2^2 + 2v_0p_2 - u_0}{p_2^2+1}\right)\left(\frac{u_0p_3^2 + 2v_0p_3 - u_0}{p_3^2+1}\right). \end{aligned}$$

Similarly, $\sin\theta_i, \cos\theta_i \in \sqrt{m}\mathbb{Q}$ implies that the area of each of the isosceles wedges is rational as $A_i = 2\sin\theta_i\cos\theta_i$ so that the area of each of these quadrilaterals is rational.

On the one hand, we could consider m to be fixed, and of the correct shape, so that we can guarantee the existence of a solution to the equation $m = u^2 + v^2$. It turns out that finding a particular solution to a general conic was originally achieved by Legendre with a descent type argument (see [4, pp. 238-239]). This is an exponential time algorithm which has been considerably improved by Cremona and Rusin (see [7]) and more recently Simon ([19]), using lattice basis reduction techniques, into an almost linear time algorithm, however they still require the factorization of m to be known—a potential bottleneck. If one does not have this factorization then one needs to resort to Cornacchia's method (see [12, Ch. 2]) which essentially relies on continued fractions.

On the other hand, if we do not care about the order that the quadrilaterals are produced, then we can consider u_0 and v_0 to be two extra free parameters—used to produce all Brahmagupta quadrilaterals. If we have two different representatives for the same m then they will produce the same quadrilaterals. Suppose that $m = u_0^2 + v_0^2 = u_1^2 + v_1^2$ then one of the rational sides can be represented in two ways, namely

$$a = \frac{2(u_0 p^2 + 2v_0 p - u_0)}{p^2 + 1} = \frac{2(u_1 q^2 + 2v_1 q - u_1)}{q^2 + 1}$$

for some rational values p, q if and only if

$$(2u_0 - a)p^2 + 4v_0 p - (2u_0 + a) = 0 \quad \text{and}$$
$$(2u_1 - a)q^2 + 4v_1 q - (2u_1 + a) = 0.$$

These quadratics have rational roots if and only if the two discriminants are rational squares. However, the two discriminants are the same, as

$$\Delta_0 = (4v_0)^2 + 4(4u_0^2 - a^2) = 16m - 4a^2 = (4v_1)^2 + 4(4u_1^2 - a^2) = \Delta_1$$

thus, p is rational if and only if q is rational and we need only use one representative for m as a sum of two squares to generate all the quadrilaterals of circumradius $\sqrt{m}$.

5 Robbins Pentagons

Since Robbins' area formula was the real motivation for this section (and in fact the entire paper) we briefly restate it.

Theorem 6 *(Robbins) Consider a cyclic pentagon with sides $a_1, \ldots, a_5$, and area K_5. If $\sigma_1, \ldots, \sigma_5$ are the symmetric polynomials in the squares of the sides, $u = 16K_5^2$, $t_2 = u - 4\sigma_2 + \sigma_1^2$, $t_3 = 8\sigma_3 + \sigma_1 t_2$, $t_4 = -64\sigma_4 + t_2^2$ and $t_5 = 128\sigma_5$ then u (hence the square of the area) satisfies the degree 7 condition*

$$ut_4^3 + t_3^2 t_4^2 - 16t_3^3 t_5 - 18ut_3 t_4 t_5 - 27u^2 t_5^2 = 0.$$

Our first result is the, by now expected, one restricting the area:

Lemma 5 *Any Robbins pentagon with five integer sides has integer area.*

Proof : Substituting $K_5 = r/s$, for coprime r and s, into Theorem 6 and clearing denominators gives us

$$0 = 16r^2 T_4^3 + s^2 T_3^2 T_4^2 - 16s^8 T_3^3 t_5 - 18s^8 T_3 T_4 t_5 - 2^8 3^3 s^{10} r^4 t_5^2 \qquad (7)$$

where

$$T_3 := s^2 t_3 = 8\sigma_3 s^2 + \sigma_1(16r^2 - 4s^2\sigma_2 + s^2\sigma_1^2)$$
$$T_4 := s^4 t_4 = -64\sigma_4 s^4 + (16r^2 - 4s^2\sigma_2 + s^2\sigma_1^2)^2.$$

Since Robbins' polynomial for u is monic with integer coefficients one observes that u must be integral. Thus $4K_4 \in \mathbb{Z}$ and we have $s \mid 4$. Furthermore, we can assume without loss of generality that $2 \nmid \gcd(a_1, a_2, a_3, a_4, a_5)$, otherwise we would have $2^{2i} \mid \sigma_i$ for $i \in \{1, 2, 3, 4, 5\}$ and $2^{2i} \mid T_i$ for $i \in \{3, 4\}$. But these imply that $2 \mid r$ forcing $s = 1$ and we are done. Thus we have 2 positive cases to consider for s.

Case (i) If $s = 2$ then r is odd so we can write $r = 2R+1$. Meanwhile equation (7) quickly reveals that

$$\sigma_1^{14} \equiv 0 \pmod 2$$

which implies that $\sigma_1 \equiv 0 \pmod 2$ and hence we have either 2 or 4 odd sides. The values of $\sigma_i \pmod 4$ vary as a function of the number of odd sides so that we end up with two subcases.

(a) If there are 2 odd sides then recalling the definition of the σ_i show that there exist integers S_i such that

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) = (4S_1 + 2, 4S_2 + 1, 2S_3, 2S_4, 2S_5).$$

Substituting all this into equation (7) gives us

$$\begin{aligned} T_3 &= 2^6 S_3 + 2^5(2S_1 + 1)((4R^2 + 4R + 1) - (4S_2 + 1) + (4S_1^2 + 4S_1 + 1)) \\ &= 2^5(2K + 1) \end{aligned}$$

and

$$\begin{aligned} T_4 &= -2^{11} S_4 + 2^8((4R^2 + 4R + 1) - (4S_2 + 1) + (4S_1^2 + 4S_1 + 1))^2 \\ &= 2^8(8L + 1) \end{aligned}$$

implying that

$$\begin{aligned} 0 = &\, 2^{28}(2R + 1)^2(8L + 1)^3 + 2^{28}(2K + 1)^2(8L + 1)^2 \\ &- 2^{35}(2K + 1)^3 S_5 - 2^{30}3^2(2K + 1)(8L + 1)S_5 \\ &- 2^{34}3^3(2R + 1)^4 S_5^2. \end{aligned}$$

Dividing this last equation by 2^{28} and reducing the result modulo 4 leads to the following contradiction

$$\begin{aligned} 0 &\equiv (4R^2 + 4R + 1)(8L + 1)^3 + (4K^2 + 4K + 1)(8L + 1)^2 \pmod 4 \\ &\equiv 2 \pmod 4 \end{aligned}$$

completing this subcase.

(b) If there are 4 odd sides then there exist integers S_i such that

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) = (4S_1, 2S_2, 4S_3, 4S_4 + 1, 2S_5).$$

Substituting all this into equation (7) gives us

$$\begin{aligned} T_3 &= 2^7 S_3 + 2^6 S_1((4R^2+4R+1) - 2S_2 + 4S_1^2) \\ &= 2^6 K \end{aligned}$$

and

$$\begin{aligned} T_4 &= -2^{10}(4S_4+1) + 2^8((4R^2+4R+1) - 2S_2 + 4S_1^2)^2 \\ &= 2^8(4L+1) \end{aligned}$$

implying that

$$\begin{aligned} 0 &= 2^{28}(2R+1)^2(4L+1)^3 + 2^{30}K^2(4L+1)^2 \\ &\quad - 2^{38}K^3 S_5 - 2^{31}3^2 K(4L+1)S_5 \\ &\quad - 2^{34}3^3(2R+1)^4 S_5^2. \end{aligned}$$

Dividing this last equation by 2^{28} and reducing the result modulo 4 again leads to a contradiction, namely,

$$\begin{aligned} 0 &\equiv (4R^2+4R+1)(4L+1)^3 \pmod 4 \\ &\equiv 1 \pmod 4) \end{aligned}$$

completing this case.

Case (ii) As for Heron triangles the most work is required when $s = 4$. This time we note that $2 \nmid \gcd(a_1, a_2, a_3, a_4, a_5)$ implies that either an odd number of the sides are odd or 2 or 4 are odd.

(a) The three subcases including an odd number of sides force σ_1 to be odd and since r is also odd we let

$$\begin{aligned} r &= 2R+1 \\ \sigma_1 &= 2S+1 \end{aligned}$$

for integer R and S. Substituting these into the defining equations for T_3 and T_4 gives us

$$\begin{aligned} T_3 &= 2^7\sigma_3 + 2^4(2S+1)(4R^2+4R+1-4\sigma_2+4S^2+4S+1) \\ &= 2^7\sigma_3 + 2^5(2S+1)(2(R^2+R-\sigma_2+S^2+S)+1) \\ &= 2^5(2K+1) \end{aligned}$$

and

$$\begin{aligned} T_4 &= -2^{14}\sigma_4 + 2^8(4R^2+4R+1-4\sigma_2+4S^2+4S+1)^2 \\ &= -2^{14}\sigma_4 + 2^{10}(2(R^2+R-\sigma_2+S^2+S)+1)^2 \\ &= 2^{10}(4L+1). \end{aligned}$$

Substituting these into equation (7) gives

$$\begin{aligned} 0 &= 2^{30}(4R^2+4R+1)(4L+1)^3 + 2^{30}(4K^2+4K+1)(4L+1)^2 \\ &- 2^{38}(2K+1)^3\sigma_5 - 2^{35}3^2(2K+1)(4L+1)\sigma_5 \\ &- 2^{45}(2R+1)^4\sigma_5^2. \end{aligned}$$

Dividing out the factor of 2^{30} and reducing the resulting equation modulo 4 leads to $0 \equiv 2 \pmod 4$—a contradiction which completes this subcase.

(b) If there are 2 odd sides then

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) = (4S_1+2, 4S_2+1, 2S_3, 2S_4, 2S_5)$$

leads to $T_3 = 2^5(2K+1)$ and $T_4 = 2^8(8L+1)$ which when substituted into equation (7) leads to

$$0 \equiv (2R+1)^2(8L+1)^3 \pmod 4,$$

which is impossible.

(c) If there are 4 odd sides then

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) = (4S_1, 2S_2, 4S_3, 4S_4+1, 2S_5)$$

leads to $T_3 = 2^6K$ and $T_4 = 2^8(4L+1)$ which when substituted into equation (7) gives

$$0 \equiv (2R+1)^2(4L+1)^3 \pmod 4,$$

which is impossible. □

Just as in the Heron triangle and the Brahmagupta quadrilateral cases, we find that this Lemma leads to the following result.

Theorem 7 *The perimeter of an integer sided Robbins pentagon is even.*

Proof : By Lemma 5 it is clear that the area is integral so if we consider the equation satisfied by the area, K_5, from Theorem 6 modulo 2 we get

$$t_3^2 t_4^2 \equiv 0 \pmod 2.$$

Substituting for t_3, t_4 and the resulting t_2 gives

$$\begin{aligned} \sigma_1^{14} &\equiv 0 \pmod 2 \quad \text{or} \\ a_1^2+a_2^2+a_3^2+a_4^2+a_5^2 &\equiv 0 \pmod 2 \quad \text{or} \\ a_1+a_2+a_3+a_4+a_5 &\equiv 0 \pmod 2. \end{aligned}$$

□

A series of searches was initiated, using Robbins' area formula, in an attempt to find an indecomposable Robbins pentagon. The results of the exploration over all cyclic pentagons with perimeter less than 400 revealed no such beast (see Table 2). While we have been unable to prove that indecomposable Robbins 5-gons do not exist the next result is the first step in this process. This theorem is in fact a corollary of Theorem 11 (below) however we leave it here due to its simplicity.

perimeter	sides	radius	area	diagonals
68	[7,7,15,15,24]	25/2	276	[336/25,20,24,117/5,25]
72	[7,15,15,15,20]	25/2	342	[20,24,24,25,117/5]
172	[16,16,25,52,63]	65/2	1638	[2016/65,39,63,253/5,65]
176	[16,25,33,39,63]	65/2	1848	[39,52,60,60,65]
178	[9,20,20,51,78]	325/8	1332	[143/5,504/13,65,1161/25,75]
178	[16,25,25,52,60]	65/2	1884	[39,600/13,63,56,836/13]
182	[16,25,33,52,56]	65/2	2058	[39,52,323/5,60,312/5]
182	[25,25,33,39,60]	65/2	2094	[600/13,52,60,63,65]
184	[16,25,39,52,52]	65/2	2148	[39,56,65,312/5,60]
186	[25,33,33,39,56]	65/2	2268	[52,3696/65,60,323/5,837/13]
188	[25,33,39,39,52]	65/2	2358	[52,60,312/5,65,63]
218	[13,13,40,68,84]	85/2	2436	[2184/85,51,84,304/5,85]
220	[9,20,51,65,75]	325/8	2760	[143/5,65,406/5,70,78]
220	[20,20,51,51,78]	325/8	2844	[504/13,65,25806/325,75,406/5]
224	[9,20,65,65,65]	325/8	2952	[143/5,75,78,78,70]
224	[13,36,40,51,84]	85/2	2856	[805/17,68,77,75,85]
226	[20,20,51,65,70]	325/8	3108	[504/13,65,406/5,75,78]
234	[13,36,40,68,77]	85/2	3276	[805/17,68,84,75,408/5]
236	[13,40,40,68,75]	85/2	3390	[51,1200/17,84,77,1364/17]
238	[12,12,55,55,104]	325/6	2424	[7752/325,65,1232/13,371/5,100]
240	[13,40,51,68,68]	85/2	3624	[51,77,85,408/5,75]
240	[36,36,40,51,77]	85/2	3696	[5544/85,68,77,416/5,85]
242	[36,40,40,51,75]	85/2	3810	[68,1200/17,77,84,1443/17]
246	[36,40,51,51,68]	85/2	4044	[68,77,408/5,85,416/5]
256	[22,39,48,62,85]	1105/24	4056	[289/5,1305/17,442/5,1127/13,91]
266	[35,35,35,44,117]	125/2	3150	[336/5,336/5,75,11753/125,100]
278	[35,35,44,44,120]	125/2	3624	[336/5,75,10296/125,100,527/5]
292	[12,55,55,65,105]	325/6	4998	[65,1232/13,100,100,1395/13]
292	[45,45,50,50,102]	425/8	5268	[1386/17,85,1500/17,104,105]
294	[12,55,55,68,104]	325/6	5112	[65,1232/13,507/5,100,2668/25]
306	[45,50,50,76,85]	425/8	6192	[85,1500/17,102,105,104]
314	[29,29,60,60,136]	425/6	4512	[24128/425,85,1848/17,532/5,125]
318	[55,65,65,65,68]	325/6	6942	[100,104,104,105,507/5]
334	[35,35,44,100,120]	125/2	6312	[336/5,75,120,100,125]
340	[35,44,44,100,117]	125/2	6786	[75,10296/125,120,527/5,3116/25]
346	[35,44,75,75,117]	125/2	7374	[75,527/5,120,120,3116/25]
354	[35,44,75,100,100]	125/2	7962	[75,527/5,125,120,117]
370	[17,17,87,105,144]	145/2	6984	[4896/145,100,144,3237/29,145]
372	[17,24,87,100,144]	145/2	7230	[203/5,105,143,116,145]
374	[29,60,60,85,140]	425/6	8022	[85,1848/17,125,125,2405/17]
376	[17,24,87,105,143]	145/2	7476	[203/5,105,144,116,4200/29]
378	[24,24,87,100,143]	145/2	7722	[6864/145,105,143,3483/29,145]
384	[29,60,60,99,136]	425/6	8712	[85,1848/17,663/5,125,3531/25]
390	[25,25,59,136,145]	3625/48	7680	[1430/29,406/5,4375/29,2529/25,150]
398	[29,60,85,99,125]	425/6	9930	[85,125,140,136,136]

Table 2: Robbins pentagons

Theorem 8 *Any Robbins pentagon has either zero or five rational diagonals and in the latter case is diagonally decomposable.*

Proof : Consider the pentagon in Figure 7 for which the diagonals opposite sides a, b, c, d, e are labelled u_1, u_2, u_3, u_4, u_5 respectively. If the pentagon has one rational diagonal, u_4 say, then it must decompose into a triangle and a quadrilateral such that

$$K_3^2(a, b, u_4), \quad K_4^2(c, d, e, u_4), \quad K_3(a, b, u_4) + K_4(c, d, e, u_4) \in \mathbb{Q},$$

which implies that $K_3(a, b, u_4) \in \mathbb{Q}$ and $K_4(c, d, e, u_4) \in \mathbb{Q}$. But by Theorem 2 we find that $R_3(a, b, u_4) \in \mathbb{Q}$ and hence that $R_4(c, d, e, u_4) \in \mathbb{Q}$. Now since

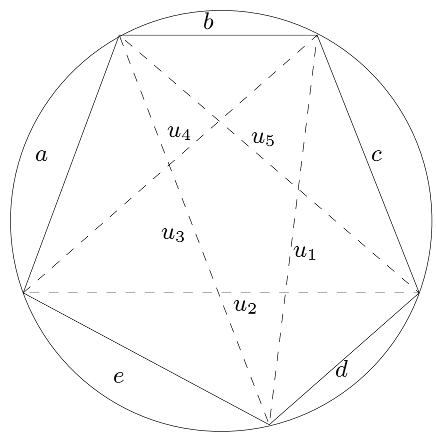

Figure 7: A cyclic pentagon

$[c, d, e, u_4]$ is radially decomposable we infer that u_1 and u_2 are rational. Similar arguments applied to other quadrilaterals reveal that all diagonals, and the areas of all sub-triangles and all sub-quadrilaterals are rational. □

We can in fact prove more, namely the equivalence of diagonal and radial decomposability.

Theorem 9 *Any Robbins pentagon is diagonally decomposable if and only if it is radially decomposable.*

Proof : If we assume diagonal decomposability then any 3 connected sides and the appropriate diagonal can be used to form a diagonally decomposable 4-gon which by Theorem 4 must be radially decomposable. Since this applies to all such sub-4-gons we see that the original 5-gon is radially decomposable.

Conversely, a single application of Lemma 3 followed by Theorem 8 forces any radially decomposable pentagon to have 5 rational diagonals and be diagonally decomposable. □

Now if we assume the extra condition that the two areas either side of a diagonal are rational (without any such constraint on the diagonal itself) then we can almost obtain the result we seek, namely that a Robbins pentagon is decomposable.

Theorem 10 *Any diagonal which decomposes a Robbins pentagon into two rational areas lies in, at worst, a quadratic extension of* $\mathbb{Q}$.

Proof : Consider again the cyclic pentagon in Figure 7. If we let K_3 denote the area of the triangle $[c, d, u_1]$ and K_4 denote the area of the quadrilateral $[a, b, e, u_1]$ then Heron's formula and Brahmagupta's formula respectively lead to

$$\begin{aligned}16K_3^2 &= -c^4 - d^4 - u_1^4 + 2c^2d^2 + 2d^2u_1^2 + 2u_1^2c^2 \\ 16K_4^2 &= (-u_1 + a + b + e)(u_1 - a + b + e)(u_1 + a - b + e)(u_1 + a + b - e).\end{aligned}$$

Rearranging these two shows that the diagonal, u_1, satisfies the two degree 4 polynomials

$$u_1^4 + Pu_1^2 + Q = 0 \quad \text{and} \quad u_1^4 + Ru_1^2 + Su_1 + T = 0$$

where

$$\begin{aligned}P &:= -2(c^2 + d^2) \\ Q &:= 16K_3^2 + (c^2 - d^2)^2 \\ R &:= -2(a^2 + b^2 + e^2) \\ S &:= -8abe \\ T &:= 16K_4^2 + (a^4 + b^4 + e^4 - 2a^2b^2 - 2b^2e^2 - 2e^2a^2)\end{aligned}$$

and $P, Q, R, S, T \in \mathbb{Q}$. Clearly u_1 must be a solution to the difference between these two polynomials and so satisfies the pair

$$\begin{aligned}(P - R)u_1^2 - Su_1 + (Q - T) &= 0 \\ u_1^4 + Ru_1^2 + Su_1 + T &= 0.\end{aligned}$$

We can assume that $P - R \neq 0$ otherwise the first equation would immediately imply that u_1 is rational. The remainder upon dividing the latter quartic by the quadratic is $\alpha u_1 + \beta$ where

$$\begin{aligned}\alpha &= S(P(P - R)^2 - 2(P - R)(Q - T) + S^2) \\ \beta &= T(P - R)^3 - (Q - T)(R(P - R)^2 - (Q - T)(P - R) + S^2).\end{aligned}$$

Since u_1 must satisfy $\alpha u_1 + \beta = 0$ then $u_1 \in \mathbb{Q}$ as long as $\alpha \neq 0$. If $\alpha = 0$ then the equation $\alpha u_1 + \beta = 0$ implies that $\beta = 0$. Now using the two conditions

$\alpha = 0 = \beta$ in the quadratic $(P - R)u_1^2 - Su_1 + (Q - T) = 0$ and eliminating Q and T reveals that u_1 satisfies the condition

$$2(P - R)^2 u_1^2 - 2S(P - R)u_1 + (P(P - R)^2 + S^2) = 0$$

Since the coefficients are only functions of the sides the result follows. □

Notice also that we could have weakened the hypothesis of the theorem and only required that $K_3^2, K_4^2 \in \mathbb{Q}$.

At this stage we are lacking a formula for the diagonal of a cyclic pentagon. It turns out that they satisfy a degree 7 polynomial. Suppose we fix on the diagonal u_1 say in Figure 7, then we can equate the circumradii of the triangle and quadrilateral either side of the diagonal to get

$$R_3(c, d, u_1) = R_4(a, b, e, u_1)$$

or

$$\frac{cdu_1}{4K_3(c, d, u_1)} = \frac{\sqrt{(au_1 + be)(ae + bu_1)(ab + eu_1)}}{4K_4(a, b, e, u_1)}$$

or

$$(au_1 + be)(ae + bu_1)(ab + eu_1)K_3^2(c, d, u_1) - (cdu_1)^2 K_4^2(a, b, e, u_1) = 0.$$

The other 5 diagonals can be obtained by simply permuting the sides or they can be combined into a single representation by labelling the edges $[a_0, a_1, a_2, a_3, a_4]$ and the corresponding diagonals as $[u_0, u_1, u_2, u_3, u_4]$ with u_i opposite a_i to get:

$$(a_i u_i + a_{i+1}a_{i+4})(a_i a_{i+4} + a_{i+1}u_i)(a_i a_{i+1} + a_{i+4}u_i)K_3^2(a_{i+2}, a_{i+3}, u_i) \\ -(a_{i+2}a_{i+3}u_i)^2 K_4^2(a_i, a_{i+1}, a_{i-1}, u_i) = 0$$

for $i = 0, 1, 2, 3, 4$ and the subscripts are taken modulo 5.

With this in hand a series of Monte Carlo tests were run in Magma to calculate the required field extension for a specific diagonal of a random cyclic pentagon. In each case the other 4 diagonals all required the same extension field of $\mathbb{Q}$ suggesting that this may well be generally true.

If we restrict back to the case of Robbins pentagons, *ie.* rational area cyclic pentagons, then we can prove the above observation.

Theorem 11 *All the diagonals of a Robbins pentagon are rational functions of any one of them hence lie in the same field as that defined by any one diagonal.*

Proof : Considering Figure 7 we let $\mathbb{Q}(u_1) = \mathbb{Q}(\zeta)$ for some element, ζ, of a number field of degree no greater than 7. Appealing to Heron and Brahmagupta again we have the following two formulæ

$$16K_3^2 = -c^4 - d^4 - u_1^4 + 2c^2d^2 + 2c^2u_1^2 + 2d^2u_1^2,$$
$$16K_4^2 = (-u_1 + a + b + e)(u_1 - a + b + e)(u_1 + a - b + e)(u_1 + a + b - e),$$

which imply that $K_3^2, K_4^2 \in \mathbb{Q}(\zeta)$. Since we also have $K_3 + K_4 \in \mathbb{Q} \subseteq \mathbb{Q}(\zeta)$ we can apply the identity

$$K_3 = \frac{(K_3 + K_4)^2 + K_3^2 - K_4^2}{2(K_3 + K_4)}$$

to obtain that $K_3, K_4 \in \mathbb{Q}(\zeta)$. Furthermore, recalling the triangle circumradius formula, we find that

$$R_3(c, d, u_1) = \frac{cdu_1}{4K_3(c, d, u_1)}$$

hence we also have the fortuitous result that $R_3(c, d, u_1) \in \mathbb{Q}(\zeta)$. Now we focus attention on another diagonal, u_3 say. Clearly

$$K_3(a, e, u_3) + K_3(b, u_3, u_1) = K_4(a, b, e, u_1)$$
$$\frac{aeu_3}{4R_3(a, e, u_3)} + \frac{bu_3u_1}{4R_3(b, u_3, u_1)} = K_4(a, b, e, u_1)$$

which upon rearrangement gives us

$$u_3 = \frac{4R_3(c, d, u_1)K_4(a, b, e, u_1)}{ae + bu_1}$$

since the circumradii are all identical. Thus $u_3 \in \mathbb{Q}(\zeta)$ as are, by iteration, all the other diagonals. □

6 Robbins Hexagons

We used Robbins' formula as our starting point and began a search for rational area cyclic hexagons.

Theorem 12 *(Robbins) Consider a cyclic hexagon with sides $a_1, \ldots, a_6$, and area K_6. If $\sigma_1, \ldots, \sigma_5$ are the symmetric polynomials in the squares of the sides, $\sigma_6' = a_1a_2a_3a_4a_5a_6$, $u = 16K_5^2$, $t_2 = u - 4\sigma_2 + \sigma_1^2$, $t_3 = 8\sigma_3 + \sigma_1 t_2 - 16\sigma_6'$, $t_4 = t_2^2 - 64\sigma_4 + 64\sigma_1\sigma_6'$ and $t_5 = 128\sigma_5 + 32t_2\sigma_6'$ then u (and hence the square of the area) satisfies either the degree 7 condition*

$$ut_4^3 + t_3^2t_4^2 - 16t_3^3t_5 - 18ut_3t_4t_5 - 27u^2t_5^2 = 0.$$

or the same condition with σ_6' replaced by its negative.

As in all previous cases so far it is sufficient to consider only even perimeters.

Lemma 6 *Every integer sided Robbins hexagon has integer area.*

Proof : A direct analog to the proof of Lemma 5. □

Theorem 13 *Every integer sided Robbins hexagon has an even perimeter.*

Proof : Use Lemma 6 and consider Robbins' formula modulo 2. □

We expected to find Eulerian examples, however, as was the case for Brahmagupta quadrilaterals, we found radially indecomposable examples as well. For example, consider the cyclic hexagon formed from the sides $[2, 5, 2, 5, 11, 5]$

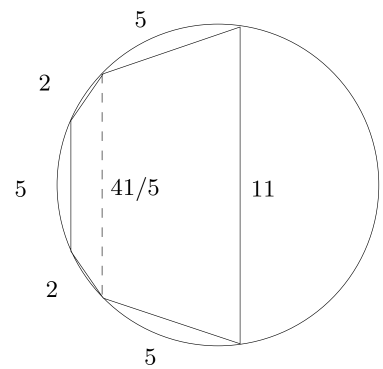

Figure 8: A radially indecomposable Robbins hexagon

as shown in Figure 8. Symmetry allows us to observe that we have two isosceles trapezia joined at the (initially unknown) central diagonal. Since Robbins' formula gives us the area of this hexagon, namely 54, we can sum two quadrilateral areas to reveal that the unknown diagonal is rational, in fact equals 41/5. Armed with this we can verify that the circumradii of the two sub-quadrilaterals (and consequently the whole hexagon) are

$$R_4(2, 5, 2, 41/5) = \frac{5\sqrt{5}}{2} = R_4(5, 11, 5, 41/5).$$

In the computational searches for Robbins 6-gons with perimeter less than 400 a total of 424 dissimilar examples have turned up. Only 3 of these have six distinct sides, while 7 are radially decomposable (see Table 3), the remainder are all non-Eulerian.

6.1 Restrictions on central diagonals

Suppose we label the vertices of a cyclic hexagon by $A_1, \ldots, A_6$ and the sides by $a_1, \ldots, a_6$ as in Figure 9. We call a diagonal *central* if it, together with the sides, splits a hexagon into two quadrilaterals, otherwise it is called a *minor* diagonal. Let D_{ij} denote the central diagonal from vertex A_i to vertex A_j, and let d_{ij} denote the minor diagonals. Then we have the following result.

sides	area	circumradius
$[7, 7, 15, 15, 15, 15]$	384	25/2
$[16, 16, 25, 25, 33, 63]$	1968	65/2
$[16, 25, 25, 25, 33, 60]$	2214	65/2
$[16, 16, 25, 25, 52, 52]$	2268	65/2
$[16, 25, 25, 33, 33, 56]$	2388	65/2
$[16, 25, 25, 33, 39, 52]$	2478	65/2
$[25, 25, 33, 33, 39, 39]$	2688	65/2
$[10, 19, 26, 40, 47, 52]$	2520	$65/8\sqrt{17}$
$[7, 14, 22, 25, 55, 73]$	2156	$5/2\sqrt{221}$
$[7, 14, 22, 25, 62, 70]$	2286	$5/2\sqrt{221}$

Table 3: A selection of Robbins hexagons

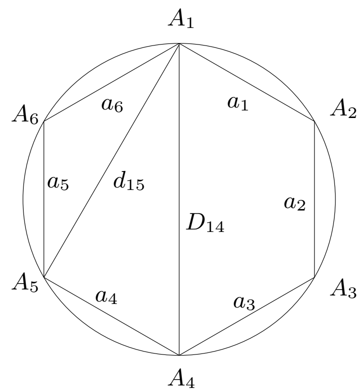

Figure 9: Diagonal constraints

Lemma 7 *If a Robbins hexagon has 1 rational central diagonal then the circumradius and all diagonals are expressible in the form $p\sqrt{m}$ for some rational p and squarefree integer m.*

Proof : First observe that a rational central diagonal and rational area force the two quadrilaterals, either side of the central diagonal to have rational area.

Suppose, without loss of generality, that $D_{14} \in \mathbb{Q}$. Then the quadrilateral $A_1A_2A_3A_4$ has a circumradius of

$$R = \frac{\sqrt{(a_1a_2 + a_3D_{14})(a_1a_3 + a_2D_{14})(a_1D_{14} + a_2a_4)}}{4K_4(a_1, a_2, a_3, D_{14})}$$

and a diagonal of

$$d_{13} = \frac{\sqrt{(a_1a_2 + a_3D_{14})(a_1a_3 + a_2D_{14})(a_1D_{14} + a_2a_4)}}{a_1a_2 + a_3D_{14}}.$$

So the squarefree portions of the radicands of R and d_{13} are the same and hence we can write $R = p_1\sqrt{m}$ and $d_{13} = p_2\sqrt{m}$. Similarly, the other diagonal of this quadrilateral is expressible as $d_{24} = p_3\sqrt{m}$.

Since the quadrilateral $A_1A_4A_5A_6$ has the same circumradius the same result applies to the two minor diagonals, $d_{15} = p_4\sqrt{m}$ and $d_{46} = p_5\sqrt{m}$.

Finally, we consider the quadrilateral $A_1A_3A_4A_5$ for which Ptolemy's theorem gives

$$a_4d_{13} + a_3d_{15} = d_{35}D_{14}$$

so that we can express the minor diagonal as

$$\begin{aligned} d_{35} &= \frac{a_4p_2\sqrt{m} + a_3p_4\sqrt{m}}{D_{14}} \\ &= p_6\sqrt{m} \end{aligned}$$

for some rational p_6. A similar argument gives $d_{26} = p_7\sqrt{m}$. □

Theorem 14 *If one central diagonal of a Robbins hexagon is rational then all three central diagonals are rational.*

Proof : Suppose $D_{14} \in \mathbb{Q}$ then applying Ptolemy's theorem to the quadrilateral $A_1A_2A_4A_5$ gives

$$D_{14}D_{25} = a_1a_4 + d_{24}d_{15}.$$

A little algebra and the proof of Lemma 7 gives

$$\begin{aligned} D_{25} &= \frac{a_1a_4 + p_3\sqrt{m}\cdot p_4\sqrt{m}}{D_{14}} \\ &= \frac{a_1a_4 + p_3p_4m}{D_{14}} \end{aligned}$$

so that D_{25} is rational. A similar argument on quadrilateral $A_1A_6A_3A_4$ leads to $D_{36} \in \mathbb{Q}$. □

6.2 Diagonal formula

Notice that we can in fact produce a formula for an arbitrary central diagonal as a function of the sides, by using an approach much like that for the diagonals of a pentagon. By equating the circumradii of the two quadrilaterals either side of a fixed central diagonal we obtain a degree seven polynomial in that diagonal.

For example, if we focus on diagonal D_{14} in Figure 9 then we obtain the polynomial

$$\begin{aligned} 0 = 4[K_4(a_4, a_5, a_6, D_{14})]^2(a_1a_2 + a_3D_{14})(a_1a_3 + a_2D_{14})(a_1D_{14} + a_2a_3) - \\ 4[K_4(a_1, a_2, a_3, D_{14})]^2(a_4a_5 + a_6D_{14})(a_4a_6 + a_5D_{14})(a_4D_{14} + a_5a_6). \end{aligned}$$

7 General cyclic polygons

During the searches for various rational area cyclic n-gons it seemed to be the case that examples were easier to come by when dealing with an even number of sides. We conjecture that the reason for this is probably that for such n-gons we have both rational circumradius examples (from Euler's generation method) as well as (quadratic) irrational circumradius examples (from our generalised Eulerian method). For an odd number of sides only rational circumradii examples seem to exist.

In this section we explore a number of different generation methods of various degrees of generality. In the end it turns out that all of them (except the first general search scheme) are subsumed by the generalised Eulerian method.

In the case of cyclic polygons with more than six sides we have no explicit area formula[3] to work with and so we are forced to use an approximate technique to discover rational area cyclic n-gons.

The process we use works as follows:

1. select the number of sides, n, and a perimeter size, p, to exhaust over,
2. select the first set of n monotonically increasing positive integers with the property that $a_i < \sum_{j \neq i} a_j$ for all $i, j \in [1..n]$ and $\sum_i a_i = p$,
3. calculate the approximate circumradius using Newton's method applied to maximising the total area of n isosceles triangles,
4. use the approximate circumradius to calculate the approximate area,
5. use continued fractions to determine if the real area looks like an integer.
6. now use a partition number enumeration algorithm to determine the next set of n sides satisfying property 2 and recurse on step 3.

So far, no rational area 7-gon has turned up, despite a search of all perimeters up to 133. Meanwhile, 6-gons, 8-gons, 10-gons and 12-gons of smaller perimeter have been found with rational area.

Once a putative area has been calculated it is usually the case that one can use a little geometry to prove that this is in fact the true area. Consider the 8-gon shown in Figure 10 which was found to have an area very close to 171. By symmetry we can arrange the sides any way we like without altering the area, so we have a rectangle surrounded by two trapezia. Clearly, the area of the 8-gon is given by

$$K_8 = 2A + B$$

where Brahmagupta's formula provides $A = (5 + L)/4\sqrt{(5 + L)(15 - L)}$ and $B = 9L$. Substituting for A, B and K_8 and squaring to remove the square root

[3] Recent work (see [11]) has produced a septagon formula, however it seems to be impractical.

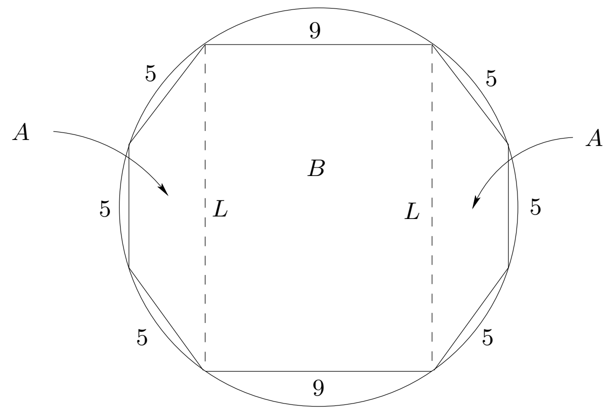

Figure 10: An integer area cyclic 8-gon

gives us

$$4(171 - 9L)^2 - (5 + L)^3(15 - L) = 0.$$

This polynomial in L has precisely one rational root, namely $L = 13$, which we subsequently use to calculate the circumradii, R_A and R_B, of the trapezia and rectangle respectively. First note that the area of one trapezium is simply $A = \frac{18\sqrt{18 \cdot 2}}{4} = 27$ so that we can use Parameśvara and Pythagoras to obtain

$$\begin{aligned} R_A &= \frac{\sqrt{(5^2 + 5 \cdot 13)(5^2 + 5 \cdot 13)(5^2 + 5 \cdot 13)}}{4 \cdot 27} \\ &= \frac{5\sqrt{10}}{2} \qquad \text{and} \\ R_B &= \frac{\sqrt{9^2 + 13^2}}{2} \\ &= \frac{5\sqrt{10}}{2}. \end{aligned}$$

In particular, the two circumradii are identical, proving that the 8-gon with sides $[5, 5, 5, 9, 5, 5, 5, 9]$ is a genuine rational area cyclic polygon.

7.1 Examples of cyclic 6-gons, 8-gons, 10-gons and 12-gons

A simple technique that we found early on allowed us to construct a limited number of rational area cyclic n-gons by tiling an isosceles quadrilateral on the 4 sides of a square. In this way we obtain a radially indecomposable 6-gon, 8-gon, 10-gon and 12-gon respectively (see Figure 11). The basic idea is to equate the circumradius of the inner square and the outer isosceles quadrilateral. The

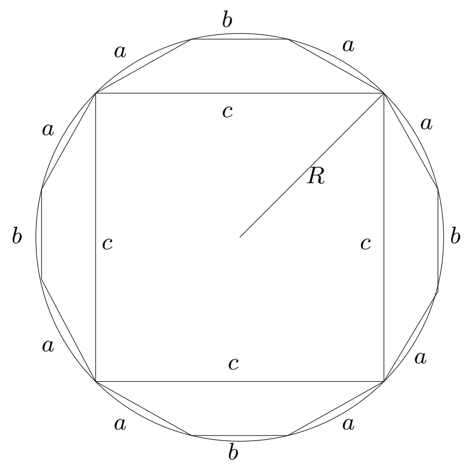

Figure 11: Cyclic rational area 6, 8, 10 and 12-gons built on a square

square has circumradius $R = c/\sqrt{2}$ while the quadrilateral has

$$R = \frac{a\sqrt{a^2 + bc}}{\sqrt{(2a - (b - c))(2a + (b - c))}}.$$

Equating these two expressions gives us a quadratic in b, namely,

$$c^2 b^2 + 2c(a^2 - c^2)b + (2a^4 - 4a^2c^2 + c^4) = 0$$

which has a discriminant of $4a^2c^2(2c^2 - a^2)$. The only way that b can be rational is for the discriminant to be a rational square, D^2 say. It is easy to show that the implied equation

$$a^2 + D^2 = 2c^2$$

has a general solution given by

$$\begin{aligned} a &= t(-r^2 - 2rs + s^2) \\ D &= t(r^2 - 2rs - s^2) \\ c &= t(r^2 + s^2). \end{aligned}$$

This general parametrization leads to a rational expression for b in terms of the free parameters r, s, t, namely,

$$\begin{aligned} b &= \frac{(c^2 - a^2) \pm a\sqrt{2c^2 - a^2}}{c} \\ &= t\frac{r^4 + 4r^3s - 6r^2s^2 - 4rs^3 + s^4}{(r^2 + s^2)}. \end{aligned}$$

Thus we get the complete rational parametrization of such n-gons as

$$a = t(-r^2 - 2rs + s^2)(r^2 + s^2)$$
$$b = t(r^4 + 4r^3s - 6r^2s^2 - 4rs^3 + s^4)$$
$$c = t(r^2 + s^2)^2.$$

When we enumerate the values of r and s the first non-trivial example we obtain occurs when $r = 1$, $s = 2$ and leads to the examples shown in Table 4.

n	perimeter	sides	area
6	102	[5,5,17,25,25,25]	688
8	104	[5,5,5,5,17,17,25,25]	751
10	106	[5,5,5,5,5,5,17,17,17,25]	814
12	108	[5,5,5,5,5,5,5,5,17,17,17,17]	877

Table 4: Specially constructed 6,8,10,12-gons

A more general search for rational area cyclic 8-gons revealed those in Table 5.

perimeter	sides	area
34	[2,2,5,5,5,5,5,5]	86
48	[5,5,5,5,5,5,9,9]	171
54	[1,1,6,6,10,10,10,10]	210
62	[5,5,5,5,8,8,13,13]	280
68	[4,4,10,10,10,10,10,10]	344
76	[5,5,5,5,11,11,17,17]	413
78	[1,1,1,15,15,15,15,15]	426
82	[5,5,5,5,5,19,19,19]	464
82	[7,7,10,10,10,10,14,14]	502
88	[3,3,11,11,15,15,15,15]	567
90	[9,9,9,10,10,10,10,23]	588
90	[5,5,5,5,14,14,21,21]	570

Table 5: Cyclic 8-gons with rational area

7.2 Isosceles quadrilateral construction

Whenever we attempt to build an n-gon with an even number of sides then we have an extra construction available that provides us with non-Eulerian examples. Basically this works by simply gluing together rescaled versions of indecomposable Brahmagupta quadrilaterals with the same circumradius. This neatly generalises the method shown in the previous section 7.1.

As a preliminary step along the way we now consider the following problem.

Find two isosceles quadrilaterals with a common circumradius and a common side.

Since this results in an important construction we outline the approach.

- First use the parametrization (6) to fix a particular isosceles Brahmagupta quadrilateral and calculate its circumradius.
- Next use one of the non-repeated sides as the common side and the fixed circumradius to solve Parameśvara's formula for remaining two sides by assuming the discriminant is a perfect square.
- Finally, select one of the infinitely many solutions which satisfies a side inequality so that it fits between the existing quadrilateral and the circle.

We work out a specific example. Suppose that $[a_1, b_1, c_1, c_1] = [9, 1, 5, 5]$ is our starting isosceles quadrilateral so that the area and circumradius are

$$K_4 = 15 \qquad \text{and} \quad R_4 = \frac{5\sqrt{34}}{6}.$$

Now we wish to find an isosceles Brahmagupta quadrilateral with sides $[a_2, b_2, c_2, c_2]$, a circumradius of $5\sqrt{34}/6$ and $b_2 = b_1 = 1$. Thus we want to solve

$$\frac{5\sqrt{34}}{6} = \frac{c_2\sqrt{c_2^2 + a_2}}{\sqrt{(2c_2 - a_2 + 1)(2c_2 + a_2 - 1)}}$$

for a_2 and c_2. Squaring this and writing it as a polynomial in a_2 leads to

$$5^2 \cdot 17a_2^2 + (18c_2^2 - 2 \cdot 5^2 \cdot 17)a_2 + (18c_2^4 - 2^2 \cdot 5^2 \cdot 17c_2^2 + 5^2 \cdot 17) = 0$$

which has solutions for rational a_2 if and only if the discriminant is a rational square. The discriminant is given by

$$D := 2^2 \cdot 29^2 c_2^2(-3^2 c_2^2 + 2 \cdot 5^2 \cdot 17)$$

so that there exists a rational E satisfying

$$E^2 = -3^2 c_2^2 + 2 \cdot 5^2 \cdot 17.$$

Now let $c_2 = 5x/3z$ and $E = 5y/z$ to transform this equation into primitive form

$$x^2 + y^2 = 34z^2$$

which has the particular solution $[x, y, z] = [3, 5, 1]$ (obtained from the starting quadrilateral) and hence the general solution

$$\frac{x}{z} = \frac{3P^2 + 10PQ - 3Q^2}{P^2 + Q^2}, \quad \frac{y}{z} = \frac{5P^2 - 6PQ - 5Q^2}{P^2 + Q^2}.$$

We want $0 < c_2 < 1$ so that $-9/2 < P/Q < 1/3$. For example, the choice $(P, Q) = (1, 4)$ leads to the solution $[a_2, b_2, c_2, c_2] = [103/17^3, 1, 25/51, 25/51]$ with area $K_4 = 317680/24137569$.

Of course we can now simply repeat the process using $103/17^3$ as the common side and so on until we have as many sides as we like.

7.3 A generalised Eulerian construction

Euler's general construction of rational circumradius n-gons (specialised to the setting of quadrilaterals in section 4.1) can be combined with the irrational circumradius construction (shown in section 4.3) to produce a single parametrisation which seems to produce all cyclic rational area n-gons.

We choose to enumerate them by circumradius, and ignore rational rescalings. So we pick an integer $m \in \mathbb{Q}^*/(\mathbb{Q}^*)^2$ together with $n-1$ free rational parameters $p_1, \ldots, p_{n-1}$ which define $n-1$ angles $\theta_1, \ldots, \theta_{n-1}$ given by

$$\sin\theta_i = \frac{u_0 p_i^2 + 2v_0 p_i - u_0}{\sqrt{m}(p_i^2+1)} \qquad \cos\theta_i = \frac{v_0 p_i^2 - 2u_0 p_i - v_0}{\sqrt{m}(p_i^2+1)}$$

where u_0 and v_0 are integers satisfying $m = u_0^2 + v_0^2$ and the angles satisfy the properties

$$0 < \theta_i \le \frac{\pi}{2}, \qquad \text{and} \qquad \sum_{i=1}^{n-1} \theta_i < \pi.$$

These angles correspond to $n-1$ sides of a cyclic n-gon with circumradius $R = \sqrt{m}$ since

$$a_i = 2\sqrt{m}\sin\theta_i = \frac{2(u_0 p_i^2 + 2v_0 p_i - u_0)}{p_i^2+1}.$$

Define the remaining half-angle, θ_n via $\theta_n = \pi - \sum_{i=1}^{n-1}\theta_i$ and then repeated application of the angle doubling trigonometric identities shows that the final side $a_n = 2\sqrt{m}\sin\theta_n$ is rational as is the corresponding wedge area. Hence the resulting n-gon has rational area and rational sides.

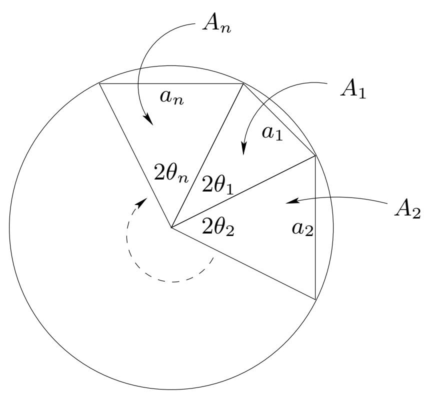

Figure 12: A general cyclic n-gon

When $m = 1$ (i.e. $u_0 = 0$ or $v_0 = 0$) we recover Euler's original parametrization, while for $m > 1$ we have a raft of new Robbins' n-gons.

7.4 The General Conjectures

If we refer back to Figure 12 and assume that the sides a_i and total area A are rational then it is easy to show that

$$A_i \in \mathbb{Q} \text{ for all } i \text{ iff } R \in \sqrt{m}\mathbb{Q}.$$

It is also clear that if any single wedge area is rational then we immediately have $R \in \sqrt{m}\mathbb{Q}$ implying that all wedges are rational.

Conjecture 1 *All cyclic n-gons with an odd number of sides and rational area are radially decomposable (hence Eulerian).*

Conjecture 2 *All cyclic n-gons with an even number of sides and rational area are either radially decomposable or quadrilaterally decomposable.*

Acknowledgement

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