The status of D21 — the search for a perfect triangle

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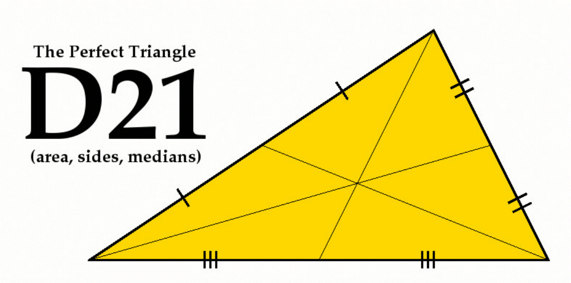
Is there a triangle with integer sides, medians and area?
Richard K. Guy (1982)



Triangles with three *rational* sides, a, b, c, three *rational* medians, k, l, m and *rational* area, Δ, satisfy the equations of

Apollonius of Perga :

$$\begin{aligned} 4k^2 &= -a^2 + 2b^2 + 2c^2, \\ 4l^2 &= -b^2 + 2c^2 + 2a^2, \\ 4m^2 &= -c^2 + 2a^2 + 2b^2, \end{aligned} \tag{1}$$

and that of Heron of Alexandria :

$$16\Delta^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c). \tag{2}$$

Defining Equations

Triangles with three *rational* sides, a, b, c, three *rational* medians, k, l, m and *rational* area, Δ, satisfy the equations of

Apollonius of Perga :

$$4(k^2, l^2, m^2) = (a^2, b^2, c^2) \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \tag{1}$$

and that of Heron of Alexandria :

$$16\Delta^2 = (a^2, b^2, c^2) \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} a^2 \\ b^2 \\ c^2 \end{pmatrix}. \tag{2}$$

Executive summary

To date no-one has either found a perfect triangle nor proven that they cannot exist—attempted proofs of impossibility, like that of Schubert [15] and others, contain flaws.

Do we search for solutions or prove they don't exist?

Search for solutions

- There exist degenerate solutions to equations (1) and (2), like $(a, b, c; k, l, m; \Delta) = (2, 4, 6; 5, 4, 1; 0)$.
- Complete solutions exist for rational sided triangles with:
  - rational area - Euler, Carmichael, B,
  - one rational median - Bachet,
  - two rational medians - Schubert, Hayes & B.
- There are infinitely many triangles which have any six of the seven quantities $(a, b, c; k, l, m; \Delta)$ rational — Euler; Rathbun, Elkies, B; Guy.

Prove they don't exist

One can eliminate triangle classes like :

- isosceles, sides in A.P., automédian, ...
- Pythagorean, Eulerian, Roberts, ...

or one can try Abel's strategy :

- find a domain in which solutions exist,
- characterise all such solutions,
- make inferences about the original problem.

Two rational medians

The Hayes parametrisation

$$
\begin{aligned}
a &= \tau\left\{(-2\phi\theta^2 - \phi^2\theta) + (2\theta\phi - \phi^2) + \theta + 1\right\} \\
b &= \tau\left\{(\phi\theta^2 + 2\phi^2\theta) + (2\theta\phi - \theta^2) - \phi + 1\right\} \\
c &= \tau\left\{(\phi\theta^2 - \phi^2\theta) + (\theta^2 + 2\theta\phi + \phi^2) + (\theta - \phi)\right\}
\end{aligned} \tag{3}
$$

where θ,ϕ and τ are arbitrary rational parameters completely describe all rational sided triangles with two rational medians

$$
\begin{aligned}
k &= \frac{\tau}{2}\left\{3\phi^2\theta + (2\theta^2 + 2\theta\phi - \phi^2) + (\theta + 2\phi) - 1\right\} \\
l &= \frac{\tau}{2}\left\{3\phi\theta^2 + (\theta^2 - 2\theta\phi - 2\phi^2) + (2\theta + \phi) + 1\right\}.
\end{aligned} \tag{4}
$$

Three rational medians

Hayes + Apollonius $\longrightarrow 4m^2 = f_4(\theta, \phi)$.

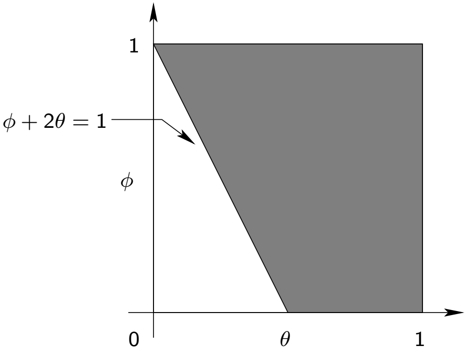


Figure: Proper triangles with three rational medians

Three rational medians

Hayes + Apollonius $\longrightarrow 4m^2 = f_4(\theta, \phi)$, which in W.N.F. is

$$E_\phi : y^2 = x^3 + A_\phi^2 x^2 - B_\phi^2 x$$

where $A_\phi = 3(1 + \phi^2)$ and $B_\phi = 24(\phi^3 - \phi)$.

- $\text{tors}(E_\phi(\mathbb{Q})) \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$ for all $\phi \in \mathbb{Q} - \{-1, 0, 1\}$. (B)
- $\text{rk}(E_\phi(\mathbb{Q})) \geq 3$ for all $\phi \in \mathbb{Q} - \left\{-1, 0, \frac{1}{2}, 1\right\}$. (Cole)
- $\text{rk}(E_{\frac{17}{70}}(\mathbb{Q})) = 7$. (B)

Hence, $\exists$ infinitely many triangles with three rational medians.

Heron and two rational medians

Hayes + Heron $\longrightarrow 16\Delta^2 = f_7(\theta, \phi)$.

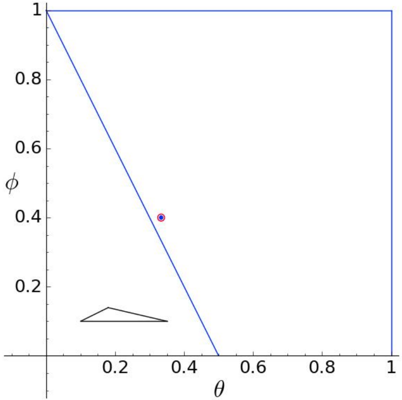


Figure: Exhaustive search (Kemnitz, Rathbun, B.)

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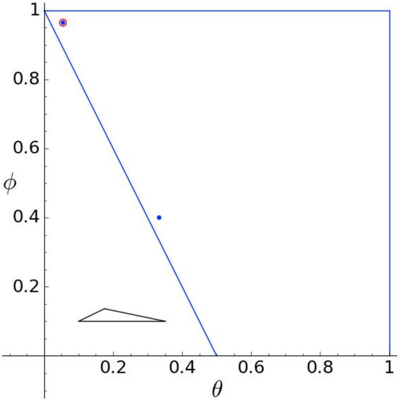

Figure: Exhaustive search (Rathbun, B.)

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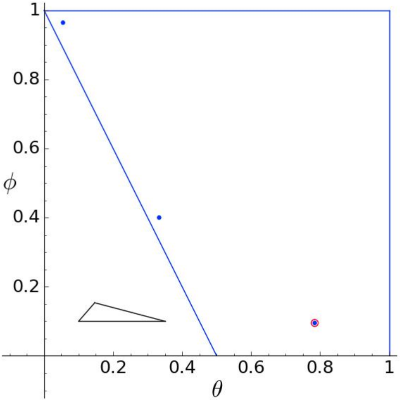

Figure: Exhaustive search (B.)

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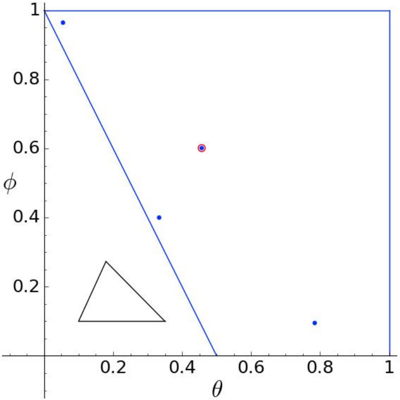

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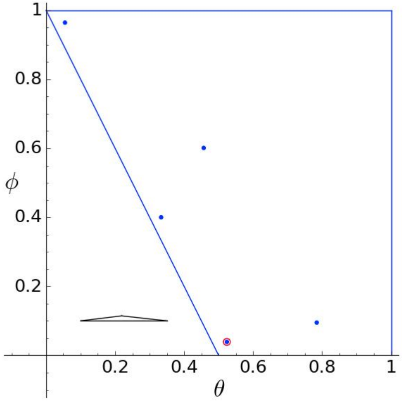

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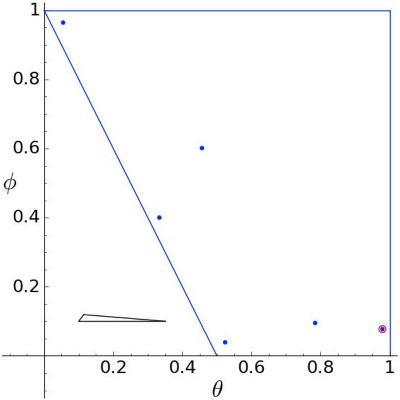

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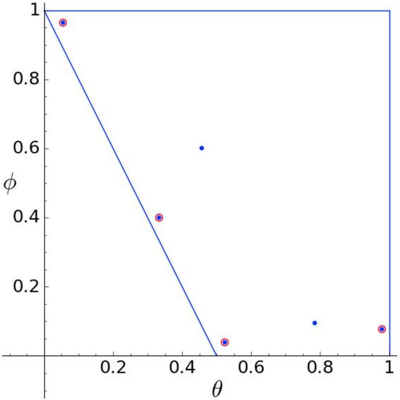

Figure: Four related H2M triangles (B)

Let

$$A[i] = \frac{A[i-1]A[i-4] + A[i-2]A[i-3]}{A[i-5]}$$

and

$$S[i] = \begin{cases} 1, 1, 2, 3, 5 & \text{for } i \in [1, 2, 3, 4, 5] \\ A[i] & \text{for } i \geq 6 \end{cases}$$

$$T[i] = \begin{cases} 1, -1, 1, 1, -7 & \text{for } i \in [1, 2, 3, 4, 5] \\ A[i] & \text{for } i \geq 6 \end{cases}$$

then Heron-2-median triangles can be generated via rational functions in $S[i], .., S[i+3]$ and $T[i], .., T[i+3]$ (Rathbun & B).

Heron and two rational medians

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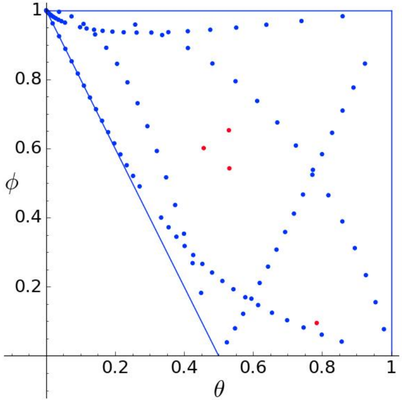


Figure: Construction via Somos sequences (Rathbun & B)

Heron and two rational medians

Hayes + Heron $\longrightarrow 16\Delta^2 = f_7(\theta, \phi)$.

- Four H2M triangles related via Schubert parameters (B 1989),
- Conjectural recursive extension (Rathbun 1989),
- Connection to two Somos sequences allowed rapid generation of 5 "apparent curves" in the $\theta\phi$-plane (Rathbun & B 1994),
- select 10 points on C_4 to obtain its equation

$$C_4 : \theta\phi(\theta - \phi) + \theta\phi + 2(\theta - \phi) - 1,$$

- All five of these curves are birationally equivalent to

$$E_{102\text{-A1}} : y^2 + xy = x^3 + x^2 - 2x.$$

Hence, $\exists$ infinitely many Heron triangles with two rational medians.

Heron and two rational medians

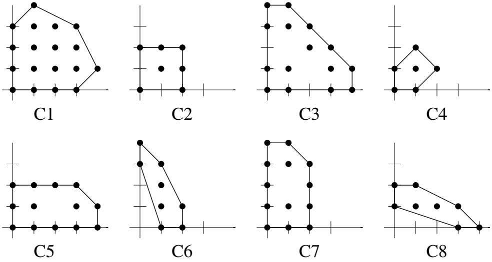
In 2003 Bácskai, Rathbun, Smith, & B. find

- a group of symmetries

$$(C_2 \times C_2)^2 \rtimes C_2$$

of the solutions to the H2M equations, and

- three more elliptic curves



birationally equivalent to E_{102-A1}.

Heron and three rational medians

Can any of the triangles coming from $C_i(\mathbb{Q})$-points have a third rational median?

- $D_i := C_i \cap \left\{4m^2 = f_4(\theta, \phi)\right\}$ are genus seven curves.
- Faltings proved they have finitely many rational points.
- Shahrina Ismail independently eliminates $\#D_4(\mathbb{Q})$-points from contention.
- $\#D_i(\mathbb{Q}) = 7, 8, 8, 10, 8, 11$ and all points lead to degenerate triangles (Stingley & B 2010).
- None of the 4 known sporadic triangles are perfect (Bácskai, Rathbun, Smith, & B 2003).

Surfaces instead of curves

Recall our four equations :

$$\begin{aligned} 4k^2 &= -a^2 + 2b^2 + 2c^2, \\ 4l^2 &= -b^2 + 2c^2 + 2a^2, \\ 4m^2 &= -c^2 + 2a^2 + 2b^2, \\ 16\Delta^2 &= (a+b+c)(-a+b+c)(a-b+c)(a+b-c). \end{aligned}$$

We now consider these as surfaces in (weighted) projective space.

Surfaces associated to D21

We study the seven projective algebraic varieties below.

	equations	ambient space	κ	sing.	surface type
S1	1-median	$\mathbb{P}^3(\mathbb{C})$	-1	0	quadric
S2	2-medians	$\mathbb{P}^4(\mathbb{C})$	-1	0	Del Pezzo
S3	3-medians	$\mathbb{P}^5(\mathbb{C})$	0	0	K3
S4	Heron	$\mathbb{P}[1^3 2^1](\mathbb{C})$	-1	6	Del Pezzo
S5	Heron-1-median	$\mathbb{P}[1^4 2^1](\mathbb{C})$	0	16	K3
S6	Heron-2-medians	$\mathbb{P}[1^5 2^1](\mathbb{C})$	2	40	Castelnuovo
S7	Heron-3-medians	$\mathbb{P}[1^6 2^1](\mathbb{C})$	2	96	general type

Six of seven

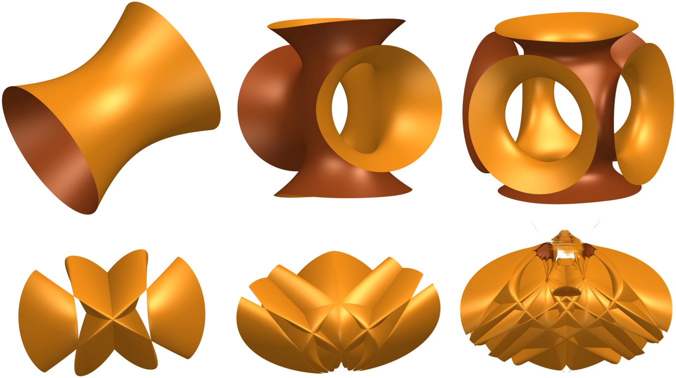


Figure: The surfaces S1, S2, S3, S4, S5, S6

Rational points on surfaces

Geometry determines arithmetic. *Hindry & Silverman.*

Q? Where are the rational points on complex algebraic surfaces?

A: It depends on the type of surface.

For example,

- $\mathbb{P}^2(\mathbb{C})$ is a rational surface.
- $V(WX - YZ) \in \mathbb{P}^3(\mathbb{C})$ is a ruled surface.
- E_ϕ is an elliptic surface.
- There exist hyperelliptic surfaces with no rational points (Skorobogatov).

Enriques-Kodaira classification

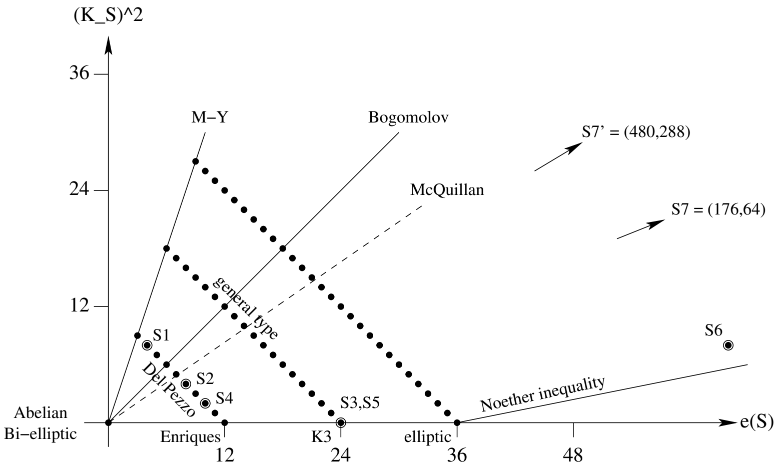


Figure: Location of eight surfaces

Rational points on surfaces

Conjecture : (Green-Griffiths, Lang) If X is a variety of general type defined over a field k then $X(k)$ is not Zariski dense in X.

This implies that any smooth algebraic surfaces of general type can only contain finitely many rational curves, elliptic curves and isolated rational points.

Theorem : (Bogomolov) Any surface X general type with $\lambda := K_S^2/e(S) > 1$ can only contain finitely many rational curves, elliptic curves and isolated rational points.

Castelnuovo surface

It appears that $S6$ contains 16 known genus 0 curves, 8 known elliptic curves and 4 known isolated points—none of which contain rational points corresponding to perfect triangles.

Can we prove that the Green-Griffiths-Lang conjecture holds for $S6$?

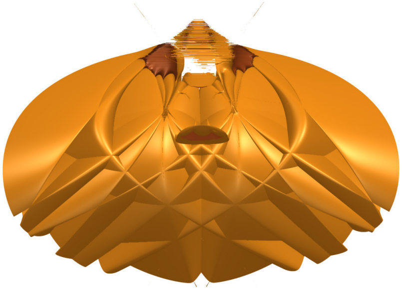


Figure: The Castelnuovo surface S6

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Code: magma, sage, surfer, apecs, ratpoints, mwrank,

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