Integrality in Algebraic Geometry[1]

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1905

[1]Translation of "Die Ganzzahligkeit in der Algebraischen Geometrie" by Ralph H. Buchholz : July 14, 2005

Contents

Introduction

Since one of the tasks of the Congress of German philologists and school teachers is the merging of science and school practice, and furthermore, since a mathematical section also belongs in such a Congress then a treatment of integrality in algebraic geometry, as contained in the following paper, would fit very well in an occasionally appearing anniversary publication[1] of a meeting of German philologists and school teachers. The representatives of mathematical science might find the research methods and the novelty of most of the results interesting, while the school teachers themselves might be interested in the results. Then they provide him the means, which allow him to specify appropriate integers for the knowns, in algebraic geometry and the geometry of solids, so that the student will also obtain integers for the unknowns.

The integrality problem is considered solved, if one succeeds in representing the knowns and the unknowns as rational numbers, and hence be recognised as commensurate. Given a group of numbers, from which every pair of ratios is rational, one can form a group of integers through multiplication by a common denominator. This technique was already used by Diophantus[2] in his arithmetic problems. Certainly Diophantus has only one pure geometric example[2] and in this uses the age old[3] realisation that strings held in a 3-4-5 triangular formation produce a right angle.

[1]In addition, the contents of this paper will appear in a book with the preliminary title "Auslese aus einer Unterrichts- und Vorlesungspraxis" (Selections from instruction and lecture notes) on which the author is still working.

[2]C.f. the following paper by Professor Büchel in Hamburg.

[3]One can read about the age of this realisation regarding the Egyptian rope stretchers in Cantor's "History of Mathematics".

1 The concept and properties of the Heron angle

To the oldest geometric integrality problems one might well include the problem of finding three integers for the three sides of a triangle such that the area is also expressible as an integer. In fact, Hero of Alexandria himself first showed how to compute the area of a triangle from the three sides, *e.g.* setting the three sides a, b, c to the value of

$$a = 13, \quad b = 14, \quad c = 15$$

leads one to the integer value of 84 for the area. Accordingly, we will begin with the assumption that sides other than these three sides of a triangle can also have integer area, and we call every such triangle, with this characteristic, Heron.

If α, β, γ are the angles of a Heron triangle, then each of the numbers

$$\tan\frac{\alpha}{2}, \quad \tan\frac{\beta}{2}, \quad \tan\frac{\gamma}{2}$$

must be rational, as is clear from the formula

$$\tan\frac{\alpha}{2} = \frac{(s-b)(s-c)}{J}$$

where J is the area of a Heron triangle and s denotes its semi-perimeter $\frac{1}{2}(a+b+c)$. So, in every Heron triangle the tangent of every half-angle must be a rational number f. From this it follows that the sine and cosine of a full Heron angle, can be set to

$$\frac{2f}{1+f^2} \quad \text{and} \quad \frac{1-f^2}{1+f^2}$$

respectively, where f is an arbitrary rational number. This is because:

$$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = 2\tan\frac{\alpha}{2}\cdot\cos^2\frac{\alpha}{2} = \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{2f}{1+f^2}$$

and

$$\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = \frac{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}} = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{1-f^2}{1+f^2}.$$

From this it also follows that:

$$\tan\alpha = \frac{2f}{1-f^2} \quad \text{and} \quad \cot\alpha = \frac{1-f^2}{2f}.$$

If we call every angle which can be an angle of a Heron triangle a Heron angle, then we can state the following theorem:

Every arbitrary rational number is equal to the tangent of half of a Heron angle. The sine (resp.) cosine of a whole Heron angle is not equal to an arbitrary rational number, but rather equal to

$$\frac{2f}{1+f^2} \quad \text{(resp.)} \quad \frac{1-f^2}{1+f^2},$$

where f denotes an arbitrary rational number.

Since every rational number is the ratio of two integers, we set

$$f = \frac{n}{m}$$

where n and m are arbitrary integers, and we recover:

$$\frac{2mn}{m^2+n^2} \quad \text{resp.} \quad \frac{m^2-n^2}{m^2+n^2}$$

for the sine and the cosine of every Heron angle. In a right-angled triangle, where the sine and cosine of an acute angle equals the ratio of two sides, we must have therefore the hypotenuse equal to $m^2 + n^2$ and each side equal to $2mn$ or $m^2 - n^2$, where m and n are integers, if this triangle is to be Heron. One can thus obtain all acute Heron angles just from the right-angled triangles, when one constructs a right-angled triangle whose sides are in the ratio

$$m^2 + n^2 \quad \text{to} \quad 2mn \quad \text{to} \quad m^2 - n^2,$$

where m and n are arbitrary positive integers and one chooses $m > n$. To find all conceivable Heron angles one has to set m and n, in

$$\sin\alpha = \frac{2mn}{m^2+n^2} \quad \text{and} \quad \cos\alpha = \frac{m^2-n^2}{m^2+n^2},$$

to all conceivable integers. But in order to generate each angle only once, one has to choose at the outset m and n without any common divisor and which are not both odd. If one does this one can find all possible Heron angles, when for each angle found one also includes its complement, or equivalently, when one sets $\sin\alpha$ not only to

$$\frac{2mn}{m^2+n^2} \quad \text{but also to} \quad \frac{m^2-n^2}{m^2+n^2}.$$

One recognises that these two forms are not substantially distinct via the following three identities:

$$2 \cdot m \cdot n = 2 \cdot \left[\left(\frac{m+n}{2} \right)^2 - \left(\frac{m-n}{2} \right)^2 \right],$$
$$m^2 - n^2 = 4 \cdot \left[\frac{m+n}{2} \cdot \frac{m-n}{2} \right],$$
$$m^2 + n^2 = 2 \cdot \left[\left(\frac{m+n}{2} \right)^2 + \left(\frac{m-n}{2} \right)^2 \right].$$

If one always chooses m greater than n then one obtains all acute Heron angles. The obtuse Heron angles are those which when added to the acute angles give 180°, or equivalently, those obtained through interchange of m and n. Actually one would be allowed to call every angle Heron, when one can arbitrarily increase or decrease it by 90° leading to an acute angle whose sine and cosine are

$$\frac{2mn}{m^2+n^2} \quad \text{and} \quad \frac{m^2-n^2}{m^2+n^2}$$

respectively, where m and n are integers. To obtain every acute Heron angle from this expression exactly once, one has to ensure that m and n have no divisor in common. In this manner, it is not necessary to express the angle itself in degrees and minutes, rather it is sufficient to know their sine and cosine, which must both be rational. Thus the search for acute Heron angles is identical to the search for Pythagorean numbers[4], that is, all integer triples, which satisfy the equation:

$$x^2 = y^2 + z^2.$$

In the sequel we will not specify the sine and cosine of the acute Heron angle, rather the sizes $m^2 + n^2$, $2mn$, $m^2 - n^2$, that are formed from the above description, where we leave it to the reader to recognise the sine and cosine of two Heron angles in each such group of numbers, which sum to 90°.

This table, which is identical to the old table of Pythagorean integers and can by be extended arbitrarily far[5], is offered to the reader as an example of Heron angles. So, *e.g.* , one should consider the Heron angle at line 13 to

[4] C.f., amongst others, in my "Math. Musestunden" (Leipzig, 1900), Band I, S. 106 and the subsequent pages.

[5] Translators note: The pair $(m, n) = (8, 7)$ appears to be missing from the table after line 14.

	m	n	m^2+n^2	$2mn$	m^2-n^2
1	2	1	5	4	3
2	3	2	13	12	5
3	4	1	17	8	15
4	4	3	25	24	7
5	5	2	29	20	21
6	5	4	41	40	9
7	6	1	37	12	35
8	6	5	61	60	11
9	7	2	53	28	45
10	7	4	65	56	33
11	7	6	85	84	13
12	8	1	65	16	63
13	8	3	73	48	55
14	8	5	89	80	39
15	9	2	85	36	77
16	9	4	97	72	65
17	9	8	145	144	17
18	10	1	101	20	99
19	10	3	109	60	91
20	10	7	149	140	51
21	10	9	181	180	19
22	11	2	125	44	117
23	11	4	137	88	105
24	11	6	157	132	85
25	11	8	185	176	57
26	11	10	221	220	21
27	12	1	145	24	143
28	12	5	169	120	119
29	12	7	193	168	95
30	12	11	265	264	23

Table 1: Table of acute Heron angles

be that with a sine of $\frac{48}{73}$ and a cosine of $\frac{55}{73}$, or its complement.

Next we consider what the characteristics of a Heron angle imply, namely, in accordance with the definition, that the tangent of the half-angles should be rational, while the sine and cosine of these half-angles are not necessarily rational. On the other hand, given a whole Heron angle all six trigonometric functions must be rational, thus an angle is Heron, firstly, if its sine and cosine are both rational, secondly, if also one of the two functions sine and cosine is rational while at the same time one of the two functions tangent and cotangent is rational. Of the angles with whole number degrees, those which are integer multiples of 90°, thus in particular 0°, 90°, 180°, are Heron. On the other hand 30°, 45°, 60° are not Heron angles. The most general characteristics of Heron angles lies in the following theorem:

Every integer coefficient combination of whole functions of Heron angles is a Heron angle.

This theorem follows from the fact that the sine or cosine of the sum or difference of two angles are rationally expressible in terms of the sine and cosine of this angle. In particular, the following theorem also results:

If $n-1$ out of n angles of an arbitrary n-gon are Heron then so is the n-th.

If here in particular, $n = 3$, we have the problem: if two angles of a triangle, α and β, are Heron so that

$$\sin\alpha = \frac{2f}{1+f^2}, \quad \sin\beta = \frac{2g}{1+g^2},$$

where f and g are rational, then the sine and cosine of the third angle γ, which must of course also be Heron, must be expressible in the same form, namely

$$\sin\gamma = \frac{2x}{1+x^2}, \quad \cos\gamma = \frac{1-x^2}{1+x^2}.$$

The solution of this problem arises from elementary trigonometry in the following way:

$$\begin{aligned}
\sin\gamma = \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \frac{2f}{1+f^2}\cdot\frac{1-g^2}{1+g^2}+ \\
+\frac{1-f^2}{1+f^2}\cdot\frac{2g}{1+g^2} = \frac{2f+2g-2fg^2-2f^2g}{(1+f^2)(1+g^2)} = \frac{2(f+g)(1-fg)}{(f+g)^2+(1-fg)^2}
\end{aligned}$$

$$\begin{aligned}
\cos\gamma &= -\cos(\alpha+\beta) = -\cos\alpha\cos\beta + \sin\alpha\sin\beta \\
&= -\frac{1-f^2}{1+f^2}\cdot\frac{1-g^2}{1+g^2} + \frac{2f}{1+f^2}\cdot\frac{2g}{1+g^2} \\
&= \frac{-1+f^2+g^2-f^2g^2+4fg}{(1+f^2)(1+g^2)} = \frac{(f+g)^2-(1-fg)^2}{(f+g)^2+(1-fg)^2}.
\end{aligned}$$

If we divide both the numerator and denominator of $\sin\gamma$ and $\cos\gamma$ by $(f+g)^2$ then we obtain $\sin\gamma$ and $\cos\gamma$ in the desired form, as long as we set

$$x = \frac{1-fg}{f+g}.$$

It is time that the rational number f, through which the sine and cosine of a Heron angle can be expressed, had a particular name. We call f the *constituent* of the Heron angle α, if

$$\sin\alpha = \frac{2f}{1+f^2} \quad \text{and} \quad \cos\alpha = \frac{1-f^2}{1+f^2}$$

and can then express our result as follows:

If f and g are the constituents of two angles of a Heron triangle then the constituent of the third equals $\frac{1-fg}{f+g}$.

Since the constituent of a Heron angle is nothing other than the tangent of the half-angle then the above result can also be derived somewhat more economically via:

$$\begin{aligned}\tan\frac{\gamma}{2} &= \tan\left(90 - \frac{\alpha+\beta}{2}\right) = \cot\frac{\alpha+\beta}{2} = \frac{1}{\tan\frac{\alpha+\beta}{2}} \\ &= \frac{1-\tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2}}{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}} = \frac{1-fg}{f+g}.\end{aligned}$$

One can also easily find the constituent, x, of the sum of n angles from the constituents of these angles

$$f_1, \quad f_2, \quad f_3, \quad \ldots \quad , \quad f_n$$

by means of the formula for

$$\tan(\alpha_1+\alpha_2+\alpha_3+\ldots+\alpha_n).$$

One obtains:

$$x = \frac{(f_1+f_2+\ldots+f_n)-(f_1f_2f_3+\ldots)+\ldots}{1-(f_1f_2+\ldots)+(f_1f_2f_3f_4+\ldots)-\ldots}.$$

If the constituent f of the angle α of a Heron triangle equals $\frac{n}{m}$, the constituent g of angle β belonging to the same triangle equals $\frac{q}{p}$, then one obtains for the constituent of the third angle γ:

$$\frac{1-fg}{f+g} = \frac{1-\frac{n}{m}\cdot\frac{q}{p}}{\frac{n}{m}+\frac{q}{p}} = \frac{mp-nq}{np+mq}.$$

If thus in a Heron triangle:

$$\sin\alpha = \frac{2mn}{m^2+n^2} \quad \text{and} \quad \sin\beta = \frac{2pq}{p^2+q^2}$$

then one has for $\sin\gamma$:

$$\sin\gamma = \frac{2\cdot\frac{mp-nq}{np+mq}}{1+\left(\frac{mp-nq}{np+mq}\right)^2} = \frac{2(mq+np)(mp-nq)}{(mq+np)^2+(mp-nq)^2}.$$

We obtain $\cos\gamma$ in a similar way:

$$\cos\gamma = \frac{1-\left(\frac{mp-nq}{np+mq}\right)^2}{1+\left(\frac{mp-nq}{np+mq}\right)^2} = \frac{(mq+np)^2-(mp-nq)^2}{(mq+np)^2+(mp-nq)^2}.$$

One finds from this, that γ is an acute, right, or obtuse angle of a triangle, depending on whether both presupposed acute angles of a triangle with constituents $\frac{n}{m}$ and $\frac{q}{p}$ satisfy the condition that,

$$\frac{mp-nq}{mq+np}$$

is smaller, equal to, or greater than one, *ie.* if respectively $\frac{p}{q}$ is smaller, equal to, or greater than

$$\frac{m+n}{m-n}.$$

If one *e.g.* attempts to produce a Heron triangle by choosing $\frac{m}{n} = \frac{2}{1}$ as the constituent of the angle α, then one knows that one obtains an acute, right, or obtuse angle corresponding to the respective choices

$$\frac{p}{q} < \frac{2+1}{2-1}, \quad \text{thus} \quad < \frac{3}{1} \quad \text{or} \quad \frac{p}{q} = \frac{3}{1} \quad \text{or} \quad \frac{p}{q} > \frac{3}{1}.$$

The choice $p = 3$, $q = 2$ leads to an acute angled triangle, the choice $p = 4$, $q = 1$ on the other hand leads to an obtuse triangle.

As already mentioned above, one obtains from the previous table all acute Heron angles, when one infers two angles from each line, namely the angle whose sine is $\frac{2mn}{m^2+n^2}$, and also its complement, *ie.* the angle whose sine is $\frac{m^2-n^2}{m^2+n^2}$. However one obtains all Heron angles if one lets the numbers m and n satisfy the condition that they are not both odd. Then one obtains every Heron angle, if one always sets $\sin\alpha$ equal to

$$\frac{2mn}{m^2+n^2}$$

and not also equal to $\frac{m^2-n^2}{m^2+n^2}$. Indeed, if m and n are both odd then $m+n$ and $m-n$ are both even, thus $\frac{m+n}{2}$ and $\frac{m-n}{2}$ are integers, hence from a regularly ordered table, one can develop the angle whose sine equals

$$\frac{2\frac{m+n}{2}\cdot\frac{m-n}{2}}{\left(\frac{m+n}{2}\right)^2+\left(\frac{m-n}{2}\right)^2}$$

or equals

$$\frac{m^2-n^2}{m^2+n^2}.$$

For this reason we will thus usually set the sine of a Heron angle to

$$\frac{2mn}{m^2+n^2}$$

but then also ensure that m and n satisfy the condition that they are not both odd.

2 Heron triangles

In §1 we observed that in a Heron triangle, *ie.* a triangle which apart from having rational sides a, b, c also has rational area J, it is the case that

$$\sin\alpha=\frac{2mn}{m^2+n^2},\quad \sin\beta=\frac{2pq}{p^2+q^2}$$
$$\sin\gamma=\frac{2(mq+np)(mp-nq)}{(m^2+n^2)(p^2+q^2)}$$

where m, n, p, q are arbitrary integers without common factors, and we suppose that $m>n$ and $p>q$ so that α and β become acute. Since:

$$a=2r\sin\alpha,\quad b=2r\sin\beta,\quad c=2r\sin\gamma$$

then we can set

$$4r=(m^2+n^2)(p^2+q^2)$$

through which we obtain

$$a=mn(p^2+q^2),\quad b=pq(m^2+n^2),\quad c=(mq+np)(mp-nq).$$

Indeed, one obtains all conceivable Heron triangles, when one sets m, n, p, q to all possible integers. One recognises that J becomes rational from:

$$J=\sqrt{s(s-a)(s-b)(s-c)}.$$

Then it is the case that:

$$s = mp(mq+np), \qquad s-a = mq(mp-nq),$$
$$s-b = np(mq-nq), \qquad s-c = nq(mq+np),$$

so that:

$$J = mnpq(mq+np)(mp-nq).$$

Right-angled Heron triangles, which one also calls Pythagorean triangles, have already been presented in the table in §1. Isosceles Heron triangles can be excluded here since they can be developed from the duplication of right-angled triangles. Isosceles triangles in which $a = b$ can be excluded as long as one does not set $p = m$ and at the same time $q = n$. On the other hand it is possible that one could have $c = a$ or $c = b$. c equals a if:

$$(mq+np)(mp-nq) = mn(p^2+q^2)$$

ie. if

$$p = 2mn \quad \text{and at the same time} \quad q = m^2 - n^2.$$

Hence one can exclude isosceles triangles as long as one avoids choosing

$$\begin{bmatrix} p = 2mn \\ q = m^2 - n^2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} m = 2pq \\ n = p^2 - q^2 \end{bmatrix}$$

for m, n, p, q. As already discussed in §1 the angle γ of a Heron triangle becomes acute or obtuse depending on whether one chooses $\frac{p}{q}$ to be less than or greater than

$$\frac{m+n}{m-n}.$$

The right-angled triangles can be avoided as long as one does not set $\frac{p}{q}$ equal to $\frac{m+n}{m-n}$. Even if one chooses m, n, p, q without common divisors, it is still possible that the resulting integers a, b, c still contain a common divisor other than two. If one chooses *e.g.* $m = 2$, $n = 1$, $p = 7$, $q = 4$, then one obtains:

$$a = 2 \cdot 65, \quad b = 2 \cdot 70, \quad c = 2 \cdot 75.$$

If one divides by two as well as by five one obtains:

$$a = 13, \quad b = 14, \quad c = 15,$$

the same triple that one obtains if one sets $m = 2$, $n = 1$, $p = 3$, $q = 2$, except that b and c appear interchanged. In the following table of all Heron triangles one finds the three indicated integers a, b, c without common factors, and if the same triple occurs twice then only the first is shown. The list is built systematically, in that, it is extended sufficiently far so as to include all possible Heron triangles, with sides which are relatively prime integers smaller than an arbitrarily given upper bound. This list, which must be completed, ought to be suitable for use in the instruction of Heron's triangle formula:

$$J = \sqrt{s(s-a)(s-b)(s-c)},$$

by a teacher wishing to specify three integers to pupils who want to procure an integral area, or equivalently, find that the root for J comes out (integrally). Right-angled and isosceles triangles have been excluded from this list due to the reasons given above. Furthermore, of any two triples which are differentiated just by the interchange of sides, only one such is included.

Since the angles α, β, γ in a Heron triangle are Heron, *ie.* they are obtained in such a way that their sines and cosines are rational, so it is the case that all the lengths of such a triangle, which are expressible in terms of the circumradius r and the trigonometric functions of the three angles, must be rational. In particular the following:

1. the three altitudes h_a, h_b, h_c, because: $h_a = 2r\sin\beta\sin\gamma$ etc.;

2. the segments in which each altitude is divided by the orthocentre, because: $AH = 2r\cos\alpha$, $HD = 2r\cos\beta\cos\gamma$ etc., where the vertices of the triangle are denoted A, B, C, the intersection of the three altitudes (orthocentre) is H and the foot of the altitude h_a is D;

3. the distances from the sides to the circumcentre; because $OD' = r\cos\alpha$, etc., where O is the circumcentre and D' the foot of the perpendicular segment from O to BC;

4. the sides a', b', c' of the (orthic) triangle constructed from the feet of the altitudes; since $a' = 2r\sin\alpha\cos\alpha$ etc.;

5. the radii ρ, ρ_a, ρ_b, ρ_c of the four circles tangent to either the sides or their extensions; because

$$\rho = \frac{2r\sin\alpha\sin\beta\sin\gamma}{\sin\alpha+\sin\beta+\sin\gamma}; \quad \rho_a = \frac{2r\sin\alpha\sin\beta\sin\gamma}{-\sin\alpha+\sin\beta+\sin\gamma} \quad \text{etc.;}$$

6. the segments on the altitudes of the orthic-triangle from the orthocentre H to the base points D'', E'', F'' on the sides mentioned in 4;

	m	n	p	q	a	b	c	J
1	2	1	3	2	13	15	14	84
2	2	1	4	1	17	10	21	84
3	2	1	5	1	52	25	63	630
4	2	1	5	2	29	25	36	360
5	2	1	5	3	68	75	77	2310
6	2	1	5	4	41	50	39	780
7	2	1	6	1	37	15	44	264
8	2	1	6	5	61	75	56	1680
9	2	1	7	1	100	35	117	1638
10	2	1	7	2	53	35	66	924
11	2	1	7	3	116	105	143	6006
12	2	1	7	5	148	175	153	10710
13	2	1	7	6	85	105	76	3192
14	2	1	8	1	13	4	15	24
15	2	1	8	3	73	60	91	2184
16	2	1	8	5	89	100	99	3960
17	2	1	8	7	113	140	99	5544
18	3	1	3	2	13	20	21	126
19	3	1	4	1	51	40	77	924
20	3	1	4	3	25	40	39	468
21	3	1	5	1	39	25	56	420
22	3	1	5	2	87	100	143	4290
23	3	1	5	3	17	25	28	210
24	3	1	5	4	123	200	187	11220
25	3	2	4	1	51	26	55	660
26	3	2	4	3	25	26	17	204
27	3	2	5	2	87	65	88	2640
28	4	1	4	3	25	51	52	624
29	4	1	5	1	104	85	171	3420
30	4	1	5	2	58	85	117	2340
31	4	3	5	1	312	125	323	19380
32	4	3	5	2	174	125	161	9660
33	5	1	5	2	29	52	69	690
34	5	1	5	3	17	39	44	330
35	5	1	5	4	41	104	105	2100
36	5	1	6	1	185	156	319	9570
..	..	..	..	..	..	..	..	..

Table 2: List of all Heron triangles

since:

$$HD'' = \frac{2r\cos\alpha\cos\beta\cos\gamma}{\cos\beta\cos\gamma + \sin\beta\sin\gamma} \quad \text{etc.}$$

The radii actually become integral if one sets $4r = (m^2+n^2)(p^2+q^2)$. On the other hand, the altitudes h_a, h_b, h_c and their segments are not necessarily integral when the sides are expressed as integers using the method given above. We provide an additional eight examples here for which all have $m = 2$, $n = 1$, thus

$$\sin\alpha = \frac{4}{5}, \quad \cos\alpha = \frac{3}{5}$$

and other than a, b, c the following integers and lengths have also been calculated:

$$\sin\beta, \quad \cos\beta, \quad \sin\gamma, \quad \cos\gamma, \quad 4r, \quad h_a, \quad h_b, \quad h_c, \quad \rho, \quad \rho_a, \quad \rho_b, \quad \rho_c.$$

These eight examples are collected in the following table. The fractions for $\sin\beta$ and for $\cos\beta$ do not need to be calculated from the values for p and q, rather can be taken from the table of Pythagorean numbers in §1. The

$\sin\beta$	$\frac{12}{13}$	$\frac{8}{17}$	$\frac{20}{29}$	$\frac{40}{41}$	$\frac{12}{37}$	$\frac{60}{61}$	$\frac{28}{53}$	$\frac{84}{85}$
$\cos\beta$	$\frac{5}{13}$	$\frac{15}{17}$	$\frac{21}{29}$	$\frac{9}{41}$	$\frac{35}{37}$	$\frac{11}{61}$	$\frac{45}{53}$	$\frac{13}{85}$
$\sin\gamma$	$\frac{56}{65}$	$\frac{84}{85}$	$\frac{144}{145}$	$\frac{156}{205}$	$\frac{176}{185}$	$\frac{224}{305}$	$\frac{264}{265}$	$\frac{304}{425}$
$\cos\gamma$	$\frac{33}{65}$	$-\frac{13}{85}$	$\frac{17}{145}$	$\frac{133}{205}$	$-\frac{57}{185}$	$\frac{207}{305}$	$-\frac{23}{265}$	$\frac{297}{425}$
$4r$	65	85	145	205	185	305	265	425
a	26	34	58	82	74	122	106	170
b	30	20	50	100	30	150	70	210
c	28	42	72	78	88	112	132	152
h_a	$\frac{336}{13}$	$\frac{336}{17}$	$\frac{1440}{29}$	$\frac{3120}{41}$	$\frac{1056}{37}$	$\frac{6720}{61}$	$\frac{3696}{53}$	$\frac{12768}{85}$
h_b	$\frac{112}{5}$	$\frac{168}{5}$	$\frac{288}{5}$	$\frac{312}{5}$	$\frac{352}{5}$	$\frac{448}{5}$	$\frac{528}{5}$	$\frac{608}{5}$
h_c	24	16	40	80	24	120	56	168
ρ	8	7	16	24	11	35	24	48
ρ_a	21	24	45	65	48	96	77	133
ρ_b	28	12	36	104	16	160	44	228
ρ_c	24	56	80	60	132	84	168	112

Table 3: Table of altitudes and tangent circle radii

examples are arranged in such a way that apart from the three sides a, b, c we also have

$$h_c, \quad \rho, \quad \rho_a, \quad \rho_b, \quad \rho_c$$

as integers.

We leave it to the reader to determine further areas within Heron triangles which are rational, like the area of certain triangles, *e.g.* the orthic-triangle, express them as rational, and likewise finally express not just the sides of a Heron triangle but also all of the related and calculated lengths and pieces of the same triangle as integers through multiplication by the common denominators.

3 Heron triangles, in which the angle bisectors are rational

If in a Heron triangle ABC the angle bisector w_a of the angle α has a rational relationship with the three sides then not only α, β, γ but also $\frac{\alpha}{2}$ must be a Heron angle. If D is the intersection of the angle-bisector with the side BC then ABD must be a Heron triangle, since w_a is rational to a, b, c, because

$$BD = \frac{a \cdot c}{b + c}$$

is rationally dependent on a, b, c. Now it is not necessarily the case that half of a Heron angle is also Heron, but its double certainly is Heron. Thus to obtain Heron triangles, in which w_a, the angle-bisector of α, is also rational we need only arrange for $\frac{\alpha}{2}$ and β to be Heron angles. Accordingly we set:

$$\sin\frac{\alpha}{2} = \frac{2uv}{u^2+v^2}, \quad \cos\frac{\alpha}{2} = \frac{u^2-v^2}{u^2+v^2},$$
$$\sin\beta = \frac{2pq}{p^2+q^2}, \quad \cos\beta = \frac{p^2-q^2}{p^2+q^2}.$$

Since $\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$ and $\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}$ we get

$$\sin\alpha = \frac{4uv(u^2-v^2)}{(u^2+v^2)^2}, \quad \cos\alpha = \frac{(u^2-v^2)^2-(2uv)^2}{(u^2+v^2)^2}$$

and, since $(u^2+v^2)^2 = (u^2-v^2)^2 + (2uv)^2$, just as in §2, we have nothing more to do than to set:

$$m = u^2 - v^2, \quad n = 2uv$$

to obtain that, other than just the rationally known lines from §2, the angle-bisector w_a is also rational. For example, suppose:

$$u = 2, \quad v = 1.$$

Then we get:

$$m = 3, \quad n = 4, \quad \text{so that} \quad \sin\alpha = \frac{24}{25}, \quad \cos\alpha = \frac{7}{25}.$$

Furthermore:

$$p = 3, \quad q = 2, \quad \text{so that} \quad \sin\beta = \frac{12}{13}, \quad \cos\beta = \frac{5}{13},$$

so that one arrives at:

$$\sin\gamma = \frac{120 + 84}{325} = \frac{204}{325}.$$

Therefore, the triangle in which the sides are as $\frac{24}{25}$ to $\frac{12}{13}$ to $\frac{204}{325}$ or as 13 times 24 to 25 times 12 to 204 or as

$$78 \quad \text{to} \quad 75 \quad \text{to} \quad 51$$

to each other must be a Heron triangle in which the angle-bisector w_a is also rational. In practice the formula expressing w_a in terms of a, b, c namely:

$$w_a = \sqrt{b \cdot c - \frac{a^2 \cdot b \cdot c}{(b+c)^2}}$$

provides a rational value for w_a if one chooses $a = 78$, $b = 75$, $c = 51$, namely:

$$w_a = \frac{340}{7}.$$

We ignore the rational expressions for the sides, the area and the angle-bisector w_a in terms of u, v, p, q, to construct several examples, in which all three angle-bisectors,

$$w_a, \quad w_b, \quad w_c$$

of a Heron triangle, are rational. Namely, if both angles $\frac{\alpha}{2}$ and $\frac{\beta}{2}$ from before are assumed to be Heron then $\frac{\gamma}{2}$, the complement of $\frac{\alpha}{2} + \frac{\beta}{2}$, must also be a Heron angle, so that all three angle-bisectors are rational. We have thus shown, that, if a triangle has rational sides, area and two rational angle-bisectors then it must also have a third rational angle-bisector. This is also evident if one multiplies:

$$w_a = \frac{2\sqrt{s(s-a)} \cdot \sqrt{bc}}{b+c}, \quad w_b = \frac{2\sqrt{s(s-b)} \cdot \sqrt{ca}}{c+a} \quad \text{and}$$
$$w_c = \frac{2\sqrt{s(s-c)} \cdot \sqrt{ab}}{a+b}$$

together. Namely, through this one obtains:

$$w_a \cdot w_b \cdot w_c = \frac{8s \cdot J \cdot a \cdot b \cdot c}{(b+c)(c+a)(a+b)},$$

where s denotes the semi-perimeter, and J the area of the triangle. We now present the calculation of four examples, in which we assume up front that $\frac{\alpha}{2}$ and $\frac{\beta}{2}$ are Heron angles and the three sides chosen so that they are composed of relatively prime integers. From this it follows that s, $s-a$, $s-b$, $s-c$, J are integers, from which furthermore it results that h_a, h_b, h_c, w_a, w_b, w_c are rational. We can calculate the angle-bisectors most easily from the formula:

$$\cos\frac{\beta-\gamma}{2} = h_a : w_a.$$

First example:

$$\cos\frac{\alpha}{2} = \frac{4}{5}, \qquad \sin\frac{\alpha}{2} = \frac{3}{5}, \qquad \cos\alpha = \frac{7}{25}, \qquad \sin\alpha = \frac{24}{25},$$
$$\cos\frac{\beta}{2} = \frac{12}{13}, \qquad \sin\frac{\beta}{2} = \frac{5}{13}, \qquad \cos\beta = \frac{119}{169}, \qquad \sin\beta = \frac{120}{169},$$

$$\cos\frac{\gamma}{2} = \sin\frac{\alpha+\beta}{2} = \frac{56}{65}, \qquad \sin\frac{\gamma}{2} = \cos\frac{\alpha+\beta}{2} = \frac{33}{65},$$
$$\cos\gamma = \frac{2047}{4225}, \qquad \sin\gamma = \frac{3696}{4225};$$
$$\cos\frac{\alpha-\beta}{2} = \frac{63}{65}; \qquad \cos\frac{\beta-\gamma}{2} = \frac{837}{845}; \qquad \cos\frac{\gamma-\alpha}{2} = \frac{323}{325}.$$

From this it follows that:

$$48r = 5^2 \cdot 13^2 = 4225; \quad \mathbf{a = 169}, \quad \mathbf{b = 125}, \quad \mathbf{c = 154};$$
$$s = 224, \quad s-a = 55, \quad s-b = 99, \quad s-c = 70, \quad \mathbf{J = 9240};$$
$$h_a = \frac{18480}{169}, \quad h_b = \frac{3696}{25}, \quad h_c = 120.$$

Finally, one obtains:

$$\mathbf{w_a} = \frac{\mathbf{30800}}{\mathbf{279}}, \quad \mathbf{w_b} = \frac{\mathbf{48048}}{\mathbf{323}}, \quad \mathbf{w_c} = \frac{\mathbf{2600}}{\mathbf{21}}.$$

Second example:

$$\cos\frac{\alpha}{2} = \frac{4}{5}, \qquad \sin\frac{\alpha}{2} = \frac{3}{5}, \qquad \cos\alpha = \frac{7}{25}, \qquad \sin\alpha = \frac{24}{25},$$
$$\cos\frac{\beta}{2} = \frac{15}{17}, \qquad \sin\frac{\beta}{2} = \frac{8}{17}, \qquad \cos\beta = \frac{161}{289}, \qquad \sin\beta = \frac{240}{289},$$
$$\cos\frac{\gamma}{2} = \frac{77}{85}, \qquad \sin\frac{\gamma}{2} = \frac{36}{85}, \qquad \cos\gamma = \frac{4633}{7225}, \qquad \sin\gamma = \frac{5544}{7225},$$

$$\cos\frac{\alpha-\beta}{2} = \frac{84}{85}; \qquad \cos\frac{\beta-\gamma}{2} = \frac{1443}{1445}; \qquad \cos\frac{\gamma-\alpha}{2} = \frac{416}{425}.$$

From this we get:

$$48r = 7225; \quad \mathbf{a} = \mathbf{289}, \quad \mathbf{b} = \mathbf{250}, \quad \mathbf{c} = \mathbf{231};$$
$$s = 385, \quad s-a = 96, \quad s-b = 135, \quad s-c = 154, \quad \mathbf{J} = \mathbf{27720};$$
$$h_a = \frac{55440}{289}, \quad h_b = \frac{5544}{25}, \quad h_c = 240.$$

Finally one obtains

$$\mathbf{w_a} = \frac{\mathbf{92400}}{\mathbf{481}}, \quad \mathbf{w_b} = \frac{\mathbf{11781}}{\mathbf{52}}, \quad \mathbf{w_c} = \frac{\mathbf{1700}}{\mathbf{7}}$$

for the three angle-bisectors.

Third example:

$$\cos\frac{\alpha}{2} = \frac{4}{5}, \qquad \sin\frac{\alpha}{2} = \frac{3}{5}, \qquad \cos\alpha = \frac{7}{25}, \qquad \sin\alpha = \frac{24}{25},$$
$$\cos\frac{\beta}{2} = \frac{60}{61}, \qquad \sin\frac{\beta}{2} = \frac{11}{61}, \qquad \cos\beta = \frac{3479}{3721}, \qquad \sin\beta = \frac{1320}{3721},$$
$$\cos\frac{\gamma}{2} = \frac{224}{305}, \qquad \sin\frac{\gamma}{2} = \frac{207}{305}, \qquad \cos\gamma = \frac{7327}{93025}, \qquad \sin\gamma = \frac{92736}{93025},$$

$$\cos\frac{\alpha-\beta}{2} = \frac{273}{305}; \qquad \cos\frac{\beta-\gamma}{2} = \frac{15717}{18605}; \qquad \cos\frac{\gamma-\alpha}{2} = \frac{1517}{1525}.$$

Now follows:

$$48r = 93025; \quad \mathbf{a} = \mathbf{3721}, \quad \mathbf{b} = \mathbf{1375}, \quad \mathbf{c} = \mathbf{3864};$$
$$s = 4480, \quad s-a = 759, \quad s-b = 3105, \quad s-c = 616, \quad \mathbf{J} = \mathbf{2550240};$$
$$h_a = \frac{5100480}{3721}, \quad h_b = \frac{92736}{25}, \quad h_c = 1320.$$

From this one obtains:

$$\mathbf{w_a} = \frac{\mathbf{8500800}}{\mathbf{5239}}, \quad \mathbf{w_b} = \frac{\mathbf{5656896}}{\mathbf{1517}}, \quad \mathbf{w_c} = \frac{\mathbf{134200}}{\mathbf{91}}$$

for the three angle-bisectors.

Fourth example:

$$\cos\frac{\alpha}{2} = \frac{12}{13}, \quad \sin\frac{\alpha}{2} = \frac{5}{13}, \quad \cos\alpha = \frac{119}{169}, \quad \sin\alpha = \frac{120}{169},$$
$$\cos\frac{\beta}{2} = \frac{21}{29}, \quad \sin\frac{\beta}{2} = \frac{20}{29}, \quad \cos\beta = \frac{48}{841}, \quad \sin\beta = \frac{840}{841},$$
$$\cos\frac{\gamma}{2} = \frac{345}{377}, \quad \sin\frac{\gamma}{2} = \frac{152}{377}, \quad \cos\gamma = \frac{95921}{142129}, \quad \sin\gamma = \frac{104880}{142129},$$

$$\cos\frac{\alpha-\beta}{2} = \frac{352}{377}; \quad \cos\frac{\beta-\gamma}{2} = \frac{10285}{10933}; \quad \cos\frac{\gamma-\alpha}{2} = \frac{4900}{4901}.$$

From this we obtain the following numbers:

$$240r = 142129, \quad \mathbf{a} = \mathbf{841}, \quad \mathbf{b} = \mathbf{1183}, \quad \mathbf{c} = \mathbf{874};$$
$$s = 1449, \quad s-a = 608, \quad s-b = 266, \quad s-c = 575, \quad \mathbf{J} = \mathbf{367080};$$
$$h_a = \frac{734160}{841}, \quad h_b = \frac{104880}{169}, \quad h_c = 840.$$

Accordingly, one obtains:

$$\mathbf{w_a} = \frac{\mathbf{1908816}}{\mathbf{2057}}, \quad \mathbf{w_b} = \frac{\mathbf{152076}}{\mathbf{245}}, \quad \mathbf{w_c} = \frac{\mathbf{39585}}{\mathbf{44}}$$

for the three angle-bisectors.

Were one to multiply the integers a, b, c found in the previous four examples by the greatest common divisor of the rational numbers

$$h_a, \quad h_b, \quad h_c, \quad w_a, \quad w_b, \quad w_c$$

one would obtain four triples of integers, which, when regarded as sizes of the sides of a triangle, would imply that not only all the altitudes but also the angle-bisectors become integral. At the same time one observes that, two arbitrarily chosen Heron angles must lead to such triples and that in a like manner there exists an infinity of similar integer triples.

Thus, by such means Heron triangles are developed, in which the half-angles of the triangle are also Heron angles, so that many other lengths, which are not necessarily rational in arbitrary Heron triangles, become rational, for example the six distances between each pair of the four centres of the circles tangent to the sides or their extensions, as well as the twelve distances of these four centres from the vertices A, B, C of the triangle.

4 Heron parallelograms

A Heron parallelogram, that is a parallelogram with rational sides rational diagonals and rational area, can contain no other Heron angle. There exists an easily derivable relationship between the angles α and β, formed at the endpoint of a diagonal which meets the two sides a and b, and the acute angle θ, formed at the intersection of the two diagonals, given by:

$$2\cot\theta = \cot\alpha - \cot\beta \tag{4.1}$$

where it is assumed that $\alpha < \beta$. From this, it follows that when the constituents of the angles α, β, θ are denoted by $\frac{n}{m}$, $\frac{q}{p}$, $\frac{y}{x}$ respectively, then

$$2\cdot\frac{x^2-y^2}{2xy} = \frac{m^2-n^2}{2mn} - \frac{p^2-q^2}{2pq}$$

or:

$$2(x+y)(x-y)mnpq = (mp+nq)(mq-np)xy. \tag{4.2}$$

Since we do not allow m and n as well as p and q and x and y to have any common factors there are only two possible ways for the existing relationship (4.2) between the ten integers:

$$x+y,\quad x-y,\quad m,\quad n,\quad p,\quad q,\quad mp+nq,\quad mp-nq,\quad x,\quad y$$

to be rationally satisfied, firstly when we set

$$x = mq,\quad y = np$$

and secondly when we set

$$x = mp,\quad y = nq.$$

In the first case one may ignore the factor $mq-np$ on both sides as $mq-np$ cannot be zero, otherwise $\frac{n}{m} = \frac{q}{p}$ or $\alpha = \beta$, which can be excluded, since in an isosceles triangle the medians to the base are identical to the altitudes to the base. Therefore one obtains:

$$2mq + 2np = mp + nq$$

or:

$$\frac{p}{q} = \frac{2m-n}{m-2n}$$

from which it follows, since $\frac{p}{q}$ and $\frac{m}{n}$ are fractions in lowest terms, that:

$$p = 2m - n \quad \text{and} \quad q = m - 2n. \tag{4.3}$$

In the second case, in which one sets $x = mp$, $y = nq$ the factor $mp+nq$ can be removed since it not allowed to be zero, otherwise

$$\tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2} = -1$$

or

$$\alpha - \beta = 180^\circ$$

would entail. Thus one has in the second case:

$$p = m + 2n \quad \text{and} \quad q = 2m + n. \tag{4.4}$$

If one denotes the constituents $\frac{n}{m}$, $\frac{q}{p}$, $\frac{y}{x}$ of the angles α, β, θ by f, g, ξ then one obtains the relationships:

$$g = \frac{1-2f}{2-f}, \quad \xi = \frac{f}{g} \tag{4.5}$$

and

$$g = \frac{2+f}{1+2f}, \quad \xi = f\cdot g. \tag{4.6}$$

in both cases.

Just as in §2 one can express the four sides, the two diagonals and the area of the parallelogram in terms of the four values:

$$m, \quad n, \quad p, \quad q$$

and hence just in terms of m and n because of the previously discovered relationships. Now one must differentiate between the diagonal e, which has angle α at one end and angle β at the other end, from the diagonal e', which lies opposite the angle $\alpha+\beta$. One can obtain the area P of the parallelogram in two ways, both through doubling the triangle with sides a, b, e and also through doubling the triangle with sides a, b, e'. One obtains from the first of the two cases:

$$\begin{aligned} a &= mn(p^2+q^2),\\ b &= pq(m^2+n^2),\\ e &= (mq+np)(mp-nq),\\ e' &= 2(m^2q^2+n^2p^2),\\ P &= 2mnpq(mq+np)(mp-nq). \end{aligned}$$

The only difference in the formulæ for the second case is that e' is not equal to $2(m^2q^2 + n^2p^2)$ but rather is equal to $2(m^2p^2 + n^2q^2)$. If one sets $p = 2m - n$, $q = m - 2n$ in the first case and $p = m + 2n$, $q = 2m + n$ in the second case, then one can express a, b, e, e', P, simply in terms of m and n. Now the formulæ for the second case are differentiated from the formulæ of the first case only by the signs of various terms, hence they can be amalgamated as follows:

$$\begin{cases} a = mn(5m^2 + 5n^2 \mp 8mn), \\ b = (m^2 + n^2)(2m^2 + 2n^2 \mp 5mn), \\ e = 2(m^2 - n^2)(m^2 \mp mn + n^2), \\ e' = 2(m^2 + n^2)^2 \mp 8mn(m^2 + n^2) + 12m^2n^2, \\ P = 4mn(2m^2 + 2n^2 \mp 5mn)(m^2 + n^2 \mp mn)(m^2 - n^2). \end{cases} \tag{4.7}$$

If one systematically substitutes here all possible pairs of integers for m and n then one obtains all possible Heron parallelograms. One should choose $m > n$ to avoid negative integers and ensure that m and n have no common divisor to avoid common divisors in a, b, e, e'. If the four integers corresponding to the sides and diagonals nevertheless have a common divisor then this is removed. Then the integer corresponding to the area P must be divided by the square of this common divisor. Samples of the formulas in (4.7), as well as the numbers in the following table satisfy the formula:

$$e^2 + e'^2 = 2a^2 + 2b^2.$$

If a pair of integers chosen for m and n lead to two groups of useful integers, then these are initially both displayed in the list of Heron parallelograms one after another; subsequently from $m = 6$ onward, one only finds the first of the two groups displayed.

5 Heron triangles with rational medians

Each of the Heron parallelograms found in §4 reveals two Heron triangles each with one rational median, because either side of both diagonals is a Heron triangle, so that half of the other diagonal is the median corresponding to that side. Alternatively, every Heron triangle with a rational median leads to a Heron parallelogram. Now it was shown in §4 (equations (4.5) and (4.6)) that Heron parallelograms are only possible when

$$g = \frac{1 - 2f}{2 - f} \quad \text{or, if} \quad g = \frac{2 + f}{1 + 2f} \tag{5.1}$$

	m	n	a	b	e	e'	P
1)	2	1	41	50	21	89	840
2)	3	1	39	25	56	34	840
3)	3	1	111	175	104	274	10920
4)	3	2	339	364	95	697	31920
5)	4	1	106	119	195	113	10920
6)	4	1	26	51	35	73	840
7)	4	3	1326	1375	259	2689	341880
8)	5	1	425	1001	744	1346	286440
9)	5	2	325	116	399	281	31920
10)	5	2	125	174	91	289	10920
11)	5	3	2175	2431	784	4546	1681680
12)	6	1	411	814	1085	697	286440
13)	7	1	679	1625	2064	1394	939120
14)	7	3	1281	319	1480	1138	341880
15)	8	3	2076	949	2695	1777	1681680
16)	9	1	1521	4879	5840	4258	6254640
..	..	..	..	..	..	..	..

Table 4: Table of Heron parallelograms

where f and g are the constituents of two angles which are formed from the diagonal which meets two sides. Thus the existence of a rational median t_c, in a Heron triangle, depends on the condition that one of the two equalities in (5.1) between the constituents f and g of the angles α and β holds. In connection with this one can also prove that a Heron triangle can only have one rational median. If there were a second rational median t_b then one must also have

$$h = \frac{1-2f}{2-f} \quad \text{or} \quad h = \frac{2+f}{1+2f} \tag{5.2}$$

where h is the constituent of the angle γ. If the triangle is Heron, one recognises that one of the equations in (5.1) is incompatible with either of the equations in (5.2), *ie.* f, g, h are rational numbers related by the equation from §1:

$$1 = fg + fh + gh. \tag{5.3}$$

From equation (5.3) it follows that:

$$h = \frac{1-fg}{f+g}$$

and from this via substitution of (5.1) we get:

$$h = \frac{2(1-f+f^2)}{1-f^2} \quad \text{or} \quad h = \frac{1-f^2}{2(1+f+f^2)}. \tag{5.4}$$

Now if both t_b and t_c are rational then one of the two equations in (5.2) must hold at the same time as one of the equations in (5.4). This would lead to one of the following equations, namely:

$$4f^3 \mp 7f^2 + 4f \mp 3 = 0 \quad \text{or} \quad 5f^3 + f = 0, \tag{5.5}$$

and actually, these must be satisfied by rational values of f so that two medians of a Heron triangle can be rational. Other than $f = 0$ there is no rational value of f which can satisfy one of the equations in (5.5). The value $f = 0$, *ie.* $\tan\frac{\alpha}{2} = 0$ leads to a degenerate Heron triangle, in which the three vertices lie on a straight line, and which indeed must also have rational medians. Outside of this borderline case there are no Heron triangles with more than one rational median. However, if one does not need to satisfy the condition that the triangle be Heron, *ie.* also have a rational area, then one can certainly find triangles with rational medians. In §10 a method is developed which produces an infinity of triangles which have three integer sides and also three integer medians.

We have already mentioned above that every Heron parallelogram leads to two Heron triangles with a rational median. One has to set either $e = c$ and $e' = t_c$ or $e' = c$ and $e = 2t_c$. In both cases one sets $P = 2J$ where J denotes the area of the Heron triangle. To obtain a, b, c, t_c as integers one always needs to multiply by two so that t_c does not become a fraction with a denominator of two. Thus one develops from the 16 examples of the table of §4 the following list of 32 Heron triangles with one rational median. For clarity each of a pair of examples which are obtained from a single example from §4 has the same number as in §4 and are discriminated from each other by a prime.

6 Heron triangles, in which a median and an angle bisector are simultaneously rational

It was shown in §3 that the rationality of an angle-bisector of a Heron triangle is tied to the condition, that not only the angle alone, from whose vertex the angle-bisector emerges, but also half of that angle is a Heron angle, that

	a	b	c	t_c	J
1)	82	100	42	89	1680
1′)	82	100	178	21	1680
2)	39	25	56	17	420
2′)	39	25	34	28	420
3)	111	175	104	137	5460
3′)	111	175	274	52	5460
4)	678	728	190	697	63840
4′)	678	728	1394	95	63840
5)	212	238	390	113	21840
5′)	212	238	226	195	21840
6)	52	102	70	73	1680
6′)	52	102	146	35	1680
7)	2652	2750	518	2689	683760
7′)	2652	2750	5378	259	683760
8)	425	1001	744	673	143220
8′)	425	1001	1346	372	143220
9)	650	232	798	281	63840
9′)	650	232	562	399	63840
10)	250	348	182	289	21840
10′)	250	348	578	91	21840
11)	2175	2431	784	2273	840840
11′)	2175	2431	4546	392	840840
12)	822	1628	2170	697	572880
12′)	822	1628	1394	1085	572880
13)	679	1625	2064	697	469560
13′)	679	1625	1394	1032	469560
14)	1281	319	1480	569	170940
14′)	1281	319	1138	740	170940
15)	4152	1898	5390	1777	3363360
15′)	4152	1898	3554	2695	3363360
16)	1521	4879	5840	2129	3127320
16′)	1521	4879	4258	2920	3127320
..	..	..	..	..	..

Table 5: List of Heron triangles with a rational median

hence w_a becomes rational if one sets

$$\sin\frac{\alpha}{2} = \frac{2uv}{u^2+v^2}, \quad \cos\frac{\alpha}{2} = \frac{u^2-v^2}{u^2+v^2}$$

where u and v are positive integers, and where we must have $v < u$ otherwise α could not be an angle of a triangle. In §4 and §5 we showed that the median

t_c becomes rational in two ways: firstly if one sets

$$p = 2m - n \quad \text{and} \quad q = m - 2n$$

where $\frac{n}{m}$ is the constituent of the angle α and $\frac{q}{p}$ is the constituent of the angle β; secondly if one sets

$$p = m + 2n \quad \text{and} \quad q = 2m + n$$

where $\frac{n}{m}$ and $\frac{q}{p}$ have the same meaning. Now when we combine the condition considered in §3 with those from §4 and §5, we obtain Heron triangles in which

$$w_a \quad \text{and} \quad t_c$$

are simultaneously rational. Since in the first case we must have $2n < m$ and because of the rationality of w_a we must get:

$$m = 2uv, \quad n = u^2 - v^2$$

and so initially we need to choose integers u and v satisfying the inequality:

$$1 < \frac{u}{v} < \frac{\sqrt{5}+1}{2} = 1.618\ldots$$

In the second case, where $q > p$, we only require that $m > n$, hence we must fulfill the condition:

$$1 < \frac{u}{v} < \sqrt{2} + 1 = 2.414\ldots$$

In both cases one obtains an infinite series of Heron triangles in which w_a and t_c are rational. From these lists, developed in such a way, we compute and print only the first example ensuring again that the three sides of the sought for Heron triangles are integers with no common factors.

Example in the first case.

The inequality

$$1 < \frac{u}{v} < \frac{\sqrt{5}+1}{2}$$

is satisfied if we assume that $u = 3$, $v = 2$. Thus one obtains successively:

$$m = 12, \quad n = 5, \quad p = 2m - n = 19, \quad q = m - 2n = 2,$$

hence:

$$\begin{aligned}
a &= {}^1/_2\, mn(p^2+q^2) = 30\cdot 365 = \mathbf{10950},\\
b &= {}^1/_2\, pq(m^2+n^2) = 19\cdot 169 = \mathbf{3211},\\
c &= {}^1/_2\, (mq+np)(mp-nq) = 119\cdot 109 = \mathbf{12971},\\
J &= {}^1/_4\, mnpq(mq+np)(mp-nq) = 5\cdot 12\cdot 19\cdot 119\cdot 109 = \mathbf{14786940},\\
2t_c &= m^2q^2+n^2p^2 = \mathbf{9601},\\
w_a &= \frac{2\sqrt{s(s-a)bc}}{b+c} = \frac{2\cdot 12\cdot 7\cdot 109\cdot 13\cdot 17\cdot 19}{16182} = \frac{\mathbf{38446044}}{\mathbf{8091}}.
\end{aligned}$$

Example in the second case.

The inequality

$$1 < \frac{u}{v} < \sqrt{2}+1$$

is satisfied if we set $u = 2$, $v = 1$. Thus one obtains:

$$m = 4, \quad n = 3, \quad p = m+2n = 10, \quad q = 2m+n = 11,$$

hence:

$$\begin{aligned}
a &= {}^1/_2\, mn(p^2+q^2) = 6\cdot 221 = \mathbf{1326},\\
b &= {}^1/_2\, pq(m^2+n^2) = 55\cdot 25 = \mathbf{1375},\\
c &= {}^1/_2\, (mq+np)(mp-nq) = 37\cdot 7 = \mathbf{259},\\
J &= {}^1/_4\, mnpq(mq+np)(mp-nq) = \mathbf{170940},\\
2t_c &= m^2q^2+n^2p^2 = \mathbf{2689},\\
w_a &= \frac{2\sqrt{s(s-a)bc}}{b+c} = \frac{\mathbf{284900}}{\mathbf{817}}.
\end{aligned}$$

7 Heron cyclic quadrilaterals

As in §2 where we used two Heron angles to construct triangles, for which the sides, the area and many other distances and areas became rational or integral, so one can use three Heron angles to construct cyclic quadrilaterals in which the sides, the diagonals and the area are expressible as integers. Namely, if

$$\alpha_1, \quad \alpha_2, \quad \alpha_3, \quad \alpha_4$$

are the four angles on the circumference of the circle opposite the four sides of the cyclic quadrilateral then we always have

$$a_i = 2r \sin \alpha_i$$

where $2r$ is the diameter of the circumcircle and:

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 180^\circ.$$

Since $\alpha_1 + \alpha_2$ and $\alpha_2 + \alpha_3$ are the peripheral angles opposite the diagonals so one can obtain rational sides and diagonals if one only assumes α_1, α_2, α_3 to be arbitrary Heron angles. Then the area J of the cyclic quadrilateral must be rational since

$$J = \frac{1}{2} ef \cdot \sin \theta = 2r^2 \sin(\alpha_1 + \alpha_2) \sin(\alpha_2 + \alpha_3) \sin(\alpha_3 + \alpha_1)$$

where e and f are the diagonals and θ is the angle constructed to give

$$\alpha_1 + \alpha_3.$$

If we set

$$\sin \alpha_1 = \frac{2p_1 q_1}{p_1^2 + q_1^2}, \quad \sin \alpha_2 = \frac{2p_2 q_2}{p_2^2 + q_2^2}, \quad \sin \alpha_3 = \frac{2p_3 q_3}{p_3^2 + q_3^2}$$

then we obtain:

$$\begin{aligned} \sin \alpha_4 &= \sin(\alpha_1 + \alpha_2 + \alpha_3) \\ &= \sin \alpha_1 \cos \alpha_2 \cos \alpha_3 + \sin \alpha_2 \cos \alpha_3 \cos \alpha_1 + \\ &\quad + \sin \alpha_3 \cos \alpha_1 \cos \alpha_2 - \sin \alpha_1 \sin \alpha_2 \sin \alpha_3. \end{aligned}$$

It is therefore obvious that we set

$$4r = (p_1^2 + q_1^2)(p_2^2 + q_2^2)(p_3^2 + q_3^2).$$

Then from the above one obtains,

$$\begin{aligned}
a_1 &= p_1q_1(p_2^2+q_2^2)(p_3^2+q_3^2),\\
a_2 &= p_2q_2(p_3^2+q_3^2)(p_1^2+q_1^2),\\
a_3 &= p_3q_3(p_1^2+q_1^2)(p_2^2+q_2^2),\\
a_4 &= p_1q_1(p_2^2-q_2^2)(p_3^2-q_3^2)+p_2q_2(p_3^2-q_3^2)(p_1^2-q_1^2)\\
&\quad + p_3q_3(p_1^2-q_1^2)(p_2^2-q_2^2)-4p_1q_1p_2q_2p_3q_3,\\
e &= (p_3^2+q_3^2)(p_1p_2-q_1q_2)(p_1q_2+p_2q_1),\\
f &= (p_1^2+q_1^2)(p_2p_3-q_2q_3)(p_2q_3+p_3q_2),\\
J &= (p_1q_2+p_2q_1)(p_2q_3+p_3q_2)(p_3q_1+p_1q_3)\\
&\qquad (p_1p_2-q_1q_2)(p_2p_3-q_2q_3)(p_3p_1-q_3q_1).
\end{aligned}$$

A confirmation is obtained for one side through use of Ptolemy's theorem for which:

$$e\cdot f = a_1\cdot a_3 + a_2\cdot a_4$$

and the other side through use of the formula:

$$J = \sqrt{(s-a_1)(s-a_2)(s-a_3)(s-a_4)},$$

which expresses the area in terms of the four sides. In reality one obtains the four equalities:

$$\begin{aligned}
s-a_1 &= (p_2q_3+p_3q_2)(p_1p_2-q_1q_2)(p_1p_3-q_1q_3),\\
s-a_2 &= (p_3q_1+p_1q_3)(p_2p_3-q_2q_3)(p_2p_1-q_2q_1),\\
s-a_3 &= (p_1q_2+p_2q_1)(p_3p_1-q_3q_1)(p_3p_2-q_3q_2),\\
s-a_4 &= (p_2q_3+p_3q_2)(p_3q_1+p_1q_3)(p_1q_2+p_2q_1),
\end{aligned}$$

whose product is the square of the expression for J given above. We provide, as an example:

$$p_1=2,\quad q_1=1,\quad p_2=3,\quad q_2=2,\quad p_3=4,\quad q_3=1.$$

Thus one obtains:

$$\begin{aligned}
a_1 &= 221,\quad a_2=255,\quad a_3=130,\quad a_4=144,\\
e &= 238,\quad f=275,\quad J=32340.
\end{aligned}$$

Concerning the four diagonal segments generated by the intersection of the two diagonals, they must at least be rational if not integral due to the sine rule because in each of the four triangles, bordered by a side and two of the diagonal segments, two, and hence all three, angles are Heron. E.g. one obtains in the chosen examples:

$$\frac{17 \cdot 48}{7} \quad \text{and} \quad \frac{17 \cdot 50}{7}$$

for the diagonal segments associated to e above, as well as:

$$\frac{5 \cdot 289}{7} \quad \text{and} \quad \frac{5 \cdot 96}{7}$$

for the diagonal segments associated to f above.

Naturally, when one extends any pair of opposite sides to their point of intersection the resulting extended lengths must be rational because of the three previously introduced Heron angles. In the following list of Heron quadrilaterals, systematically generated in order to obtain all such quadrilaterals, the six values for the four sides and two diagonals are integers without any common divisors and furthermore, by appropriate choice of p_1, q_1, p_2, q_2, p_3, q_3 there are no equal sides. On the other hand, right angles between sides of the cyclic quadrilateral are not excluded.

	1)	2)	3)		4)	5)	6)		7)	8)	
p_1	2	2	2	..	2	2	2	..	2	2	..
q_1	1	1	1	..	1	1	1	..	1	1	..
p_2	3	3	3	..	3	3	3	..	4	4	..
q_2	1	1	1	..	2	2	2	..	1	1	..
p_3	3	4	4	..	4	5	6	..	4	5	..
q_3	2	1	3	..	1	2	1	..	3	4	..
a_1	52	68	20	..	221	377	481	..	85	697	..
a_2	39	51	15	..	255	435	555	..	50	410	..
a_3	60	40	24	..	130	325	195	..	102	850	..
a_4	25	75	7	..	144	129	391	..	45	319	..
J	1764	3234	234	..	32340	88704	147840	..	4368	275184	..
e	65	85	25	..	238	406	518	..	105	861	..
f	63	77	23	..	275	440	600	..	104	840	..

Table 6: List of Heron cyclic quadrilaterals. A) $p_1 = 2, q_1 = 1$.

	9)	10)		11)		12)		13)	14)	
p_1	3	3	..	3	..	3	..	3	3	..
q_1	1	1	..	1	..	1	..	2	2	..
p_2	3	3	..	4	..	4	..	4	4	..
q_2	2	2	..	1	..	3	..	1	1	..
p_3	4	4	..	4	..	5	..	4	5	..
q_3	1	3	..	3	..	1	..	3	1	..
a_1	663	325	..	1275	..	975	..	1275	204	..
a_2	1020	500	..	1000	..	1560	..	650	104	..
a_3	520	520	..	2040	..	625	..	1326	85	..
a_4	817	19	..	1403	..	1184	..	259	195	..
J	533610	83538	..	1873872	..	1058148	..	583440	18810	..
e	1071	525	..	1925	..	1521	..	1375	220	..
f	1100	340	..	2080	..	1615	..	1352	171	..

Table 7: List of Heron cyclic quadrilaterals. B) $p_1 = 3$.

8 Heron cyclic polygons

The concept of the Heron angle, introduced in §1, led in §2 to triangles with integer sides and integer area and in §7 to cyclic quadrilaterals with integer sides, integer diagonals and integer area. This path leads further to general cyclic polygons with the same properties. Of the n peripheral angles $\alpha_1, \alpha_2, \ldots, \alpha_n$ which lie opposite the sides $a_1, a_2, \ldots, a_n$ of a cyclic polygon, if one takes $n-1$ of them to be Heron angles then since the sum of all n angles amounts to 180°, by §1 the n-th angle must also be Heron. Thus all sides must be rational since

$$a_i = 2r \sin \alpha_i$$

where $2r$ denotes the diameter of the circumcircle. Also, all possible diagonals must be rational since the relationship of every diagonal to $2r$ equals the sine of the sum of any subset of the peripheral angles α. That the area J must also be rational can be recognised from the fact that on the one hand J is the sum of the areas of many Heron triangles, and on the other hand, since:

$$J = \frac{r^2}{2}(\sin 2\alpha_1 + \sin 2\alpha_2 + \ldots + \sin 2\alpha_n)$$

must hold. If one calls the constituents (§1) of the first $n-1$ peripheral angles the arbitrarily chosen values:

$$\frac{q_1}{p_1}, \quad \frac{q_2}{p_2}, \quad \ldots, \quad \frac{q_{n-1}}{p_{n-1}},$$

then one can easily express all the sides, all the diagonals, and also the area as whole number functions of the $2(n-1)$ integers

$$q_1, \quad p_1, \quad q_2, \quad p_2, \quad q_3, \quad p_3, \quad \ldots, \quad q_{n-1}, \quad p_{n-1}$$

if one sets

$$4r = (p_1^2+q_1^2)(p_2^2+q_2^2)\ldots(p_{n-1}^2+q_{n-1}^2).$$

Imagining that all the $\frac{1}{2}n(n-3)$ diagonals have been chosen then one recognises that every resulting angle must be Heron, and that hence all diagonal segments must be rational if not also integral.

We take an example with $n=5$:

$$\frac{q_1}{p_1}=\frac{3}{1}, \quad \frac{q_2}{p_2}=\frac{5}{1}, \quad \frac{q_3}{p_3}=\frac{4}{1}, \quad \frac{q_4}{p_4}=\frac{7}{1}.$$

From this we obtain successively,

$$\begin{cases} \sin\alpha_1 = \frac{3}{5}; & \cos\alpha_1 = \frac{4}{5}; \\ \sin\alpha_2 = \frac{5}{13}; & \cos\alpha_2 = \frac{12}{13}; \\ \sin\alpha_3 = \frac{8}{17}; & \cos\alpha_3 = \frac{15}{17}; \\ \sin\alpha_4 = \frac{7}{25}; & \cos\alpha_4 = \frac{24}{25}; \end{cases} \tag{8.1}$$

$$\begin{cases} \sin(\alpha_1+\alpha_2) = \frac{56}{65}; & \cos(\alpha_1+\alpha_2) = \frac{33}{65}; \\ \sin(\alpha_1+\alpha_3) = \frac{77}{85}; & \cos(\alpha_1+\alpha_3) = \frac{36}{85}; \\ \sin(\alpha_1+\alpha_4) = \frac{4}{5}; & \cos(\alpha_1+\alpha_4) = \frac{3}{5}; \\ \sin(\alpha_2+\alpha_3) = \frac{171}{221}; & \cos(\alpha_2+\alpha_3) = \frac{140}{221}; \\ \sin(\alpha_2+\alpha_4) = \frac{204}{325}; & \cos(\alpha_2+\alpha_4) = \frac{253}{325}; \\ \sin(\alpha_3+\alpha_4) = \frac{297}{425}; & \cos(\alpha_3+\alpha_4) = \frac{304}{425}; \end{cases} \tag{8.2}$$

$$\begin{cases} \sin(\alpha_1+\alpha_2+\alpha_3) = \frac{1104}{1105}; & \cos(\alpha_1+\alpha_2+\alpha_3) = \frac{47}{1105}; \\ \sin(\alpha_1+\alpha_2+\alpha_4) = \frac{63}{65}; & \cos(\alpha_1+\alpha_2+\alpha_4) = \frac{16}{65}; \\ \sin(\alpha_1+\alpha_3+\alpha_4) = \frac{84}{85}; & \cos(\alpha_1+\alpha_3+\alpha_4) = \frac{13}{85}; \\ \sin(\alpha_2+\alpha_3+\alpha_4) = \frac{5084}{5525}; & \cos(\alpha_2+\alpha_3+\alpha_4) = \frac{2163}{5525}. \end{cases} \tag{8.3}$$

From the previously obtained values for the sine and cosine of each sum of three of five peripheral angles one can obtain the values for the sine and cosine of the sum of every four of five peripheral angles and from this also the fifth peripheral angle α_5. Namely, we obtain:

$$\sin(\alpha_1+\alpha_2+\alpha_3+\alpha_4) = \sin(180^\circ-\alpha_5) = +\sin\alpha_5 = +\frac{1073}{1105};$$
$$\cos(\alpha_1+\alpha_2+\alpha_3+\alpha_4) = \cos(180^\circ-\alpha_5) = -\cos\alpha_5 = +\frac{264}{1105}.$$

From the computed values we obtain integer values for the five sides and the five diagonals of a cyclic pentagon, if we set:

$$2r = 5525.$$

Hence one obtains the following integers for the five sides a_1, a_1, a_3, a_4, a_5 and the five diagonals d_1, d_1, d_3, d_4, d_5:

$$\begin{cases} a_1 = 3315 \\ a_2 = 2125 \\ a_3 = 2600 \\ a_4 = 1547 \\ a_5 = 5365 \end{cases} \quad \begin{cases} d_1 = 4760 \\ d_2 = 3861 \\ d_3 = 5084 \\ d_4 = 4275 \\ d_5 = 5520. \end{cases} \tag{8.4}$$

For the area J one obtains in multiple ways:

$$J = 13\ 362972.$$

In the same way, because of the sine rule, the five times three segments, into which the diagonals are divided by the other diagonals, can be rationally expressed in terms of $2r$ and the sines of the five angles or their sums. For example, one obtains for the three segments b_1, e_1, c_1 of the diagonal d_1:

$$b_1 = 2r\frac{\sin\alpha_1\cdot\sin\alpha_5}{\sin(\alpha_2+\alpha_5)}; \quad e_1 = 2r\frac{\sin\alpha_1\sin\alpha_2\sin\alpha_4}{\sin(\alpha_2+\alpha_5)\sin(\alpha_1+\alpha_2)};$$
$$c_1 = 2r\frac{\sin\alpha_2\cdot\sin\alpha_3}{\sin(\alpha_1+\alpha_3)}.$$

Subsequently the three segments corresponding to the diagonal d_1, which was equal to 4760 in our example, are:

$$\frac{91205}{28}, \quad \frac{122825}{308}, \quad \frac{91205}{28}.$$

Similarly one obtains for all 15 diagonal segments rational values which are not necessarily integral.

Since the Heron cyclic pentagon contains nothing but Heron angles then all ten triangles which lie in the pentagon are Heron. The inner pentagon whose sides are the middle segments of the diagonals has rational sides and rational area however it does not necessarily have rational diagonals. However the perpendicular heights, from the vertices of this inner pentagon to the opposite sides, are rational. Furthermore, the five triangles outside the cyclic pentagon resulting from the extension of the sides must also be Heron.

Just as a cyclic pentagon can be produced from four arbitrary Heron angles, one can construct a cyclic hexagon from five arbitrary Heron angles in which all sides and all diagonals as well as the area are rational and in fact from $n-1$ arbitrary Heron angles an n-sided cyclic polygon with the same properties. Thus the n sides, the $\frac{1}{2}n(n-3)$ diagonals, and the area, all become integral if one chooses the diameter $2r$ of the circumcircle to equal the common denominator of the $n-1$ fractions which were picked as the sines of the $n-1$ Heron angles. The parts of each diagonal are not necessarily integral, however are rational. For $n=6$ one obtains two types of diagonals, namely firstly ones that together with two hexagon edges construct a triangle, secondly those that together with three hexagon edges construct a quadrilateral. Each of the six diagonals of the first type is divided into four pieces by the other diagonals, and each of the three diagonals of the second type are divided into five pieces, so that 39 diagonal pieces result for the case $n=6$. For $n=7$ one obtains 84 diagonal segments which all become rational, as long as the six angles, from whose sines one begins, are Heron, etc. To this we also include the sides of the triangles lying outside the cyclic polygon formed from the extension of the polygon sides. These triangles as well as all the triangles inside the polygon are Heron, and thus have not only rational sides, but also rational area, rational altitudes, rational circumradius, rational incircle radii etc., as discussed in §2.

As above, in the same way as one finds completely general Heron cyclic polygons one can also find particular Heron cyclic polygons in which two or more of the sides are equal. Since the triangles were already dealt with in §2 and the cyclic quadrilaterals in §7 we begin with $n=5$.

If all five sides of a cyclic pentagon are equal then all five peripheral angles must also be equal to on another, in fact each equals $36°$. Then rationality is impossible. However, if the five sides only have two distinct lengths, so that the corresponding peripheral angles also have only two possible sizes then one can succeed in solving the related integrality problem. If we denote the different sides of the cyclic pentagon by a and b, and the corresponding opposite peripheral angles by α and β, then there are three cases to distinguish depending on the sequential ordering:

$$
\begin{array}{llllll}
1) & a, & a, & a, & a, & b; \\
2) & a, & a, & a, & b, & b; \\
3) & a, & b, & a, & b, & a.
\end{array}
$$

In the first of these three cases one obtains from $4\alpha + \beta = 180°$:

$$\sin\beta = \sin 4\alpha \quad \text{and} \quad \cos\beta = -4\cos 4\alpha.$$

We need only choose α as a Heron angle so that 4α and β become Heron angles, and thus to reach the conclusion that the four equal sides a, the fifth side b, and the two distinct diagonals e and f are integers, where e is the triply occurring diagonal which together with two copies of side a build a triangle, and f is the doubly occurring diagonal which together with a and b construct a triangle. If, for example, α is the Heron angle whose sine is $\frac{3}{5}$ and whose cosine is $\frac{4}{5}$ then when one sets:

$$2r = 625$$

and removes a common divisor of three one obtains:

$$a = 125, \quad b = 112, \quad e = 220, \quad f = 195.$$

One obtains in a number of possible ways, the area J as:

$$J = 25752.$$

In the second of the three cases, in which the Heron pentagon has three equal sides a and two equal sides b in the sequence a, a, a, b, b, we have to set:

$$3\alpha + 2\beta = 180°$$

to obtain:

$$\beta = 90° - \frac{3}{2}\alpha.$$

Accordingly, if we begin with a Heron angle $\frac{\alpha}{2}$ and obtain from this the Heron angles $\frac{3}{2}\alpha$ and β then we must obtain integer sides and integer diagonals for such pentagons if we set $2r$ equal to the numerator of the fraction obtained for $\frac{3}{2}\alpha$. Meanwhile the diagonals have three distinct lengths. The length of the diagonal e which with sides b and b construct a triangle appears once, the diagonal f which with a and a constructs a triangle appears twice, and the diagonal g which with a and b construct a triangle also appears twice. To have such an example at the outset $\frac{\alpha}{2}$ must be a Heron angle whose sine is $\frac{5}{13}$ and whose cosine is $\frac{12}{13}$. Then we obtain:

$$\sin {}^3/_2\ \alpha = \cos\beta = \frac{2035}{2197}$$
$$\cos {}^3/_2\ \alpha = \sin\beta = \frac{828}{2197}$$

so that when we set $2r = 2197$ it follows that:

$$\begin{aligned}
&a = 2r\sin\alpha = 1560, \quad b = 2r\sin\beta = 828,\\
&e = 2r\sin 2\beta = 2197\cdot\frac{2\cdot 828\cdot 2035}{2197^2} = \frac{3369960}{13^3}\\
&f = 2r\sin 2\alpha = 2197\cdot\frac{2\cdot 120\cdot 119}{169^2} = \frac{28560}{13}\\
&g = 2r\sin(\alpha+\beta) = 2197\cdot\left(\frac{120\cdot 2035}{13^5} + \frac{119\cdot 828}{13^5}\right) = 2028.
\end{aligned}$$

The integers obtained for a and b have a common factor of twelve. Dividing out by this leads to:

$$\begin{aligned}
&a = 130, \quad b = 69, \quad e = \frac{280830}{13^3},\\
&f = \frac{2380}{13}, \quad g = \frac{28561}{13^2} = 169.
\end{aligned}$$

So that the cyclic pentagon is possible the Heron angle $\frac{\alpha}{2}$, the one with which we started, must have a sine smaller than $\frac{1}{2}$.

Just as for the Heron cyclic quadrilaterals discussed in §7 not only the whole diagonals of a cyclic pentagon but also all their segments must be rational.

Thirdly, we need to discuss the cyclic pentagon in which the two different side lengths a and b lie in the sequence:

$$a, \quad b, \quad a, \quad b, \quad a.$$

Here we must distinguish between two different diagonal lengths, f and g, for which $f = 2r\sin 2\alpha$ occurs only once, $g = 2r\sin(\alpha+\beta)$ appears four times. If, as in the second case, we begin with $\sin\frac{\alpha}{2} = \frac{5}{13}$ and then obtain:

$$a = 130, \quad b = 69$$

we get in this third case:

$$f = \frac{2380}{13}, \quad g = \frac{28561}{13^2} = 169.$$

In all three cases the area is also a rational number.

Similarly to $n = 5$, where various particular cyclic polygons were treated, one can find cyclic polygons, for every arbitrary n, with rational sides, rational diagonals and rational area with the property that two or more of the sides of the cyclic polygon are equal. However, one cannot assume that each of the distinct side lengths can appear more than twice. Namely, if we denote the sides of the cyclic polygon by

$$a_1, \quad a_2, \quad a_3, \quad \ldots, \quad a_n$$

and the corresponding peripheral angles by

$$\alpha_1, \quad \alpha_2, \quad \alpha_3, \quad \ldots, \quad \alpha_n$$

and each side a_i occurs x_i times, then the relationship

$$x_1 \cdot \alpha_1 + x_2 \cdot \alpha_2 + x_3 \cdot \alpha_3 \ldots + x_n \cdot \alpha_n = 180^\circ$$

must hold, which, if $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are always supposed to be Heron angles can occur even if only one of the coefficients

$$x_1, \quad x_2, \quad x_3, \quad \ldots, \quad x_n$$

is not greater than two. Because then this one can be expressed in terms of the remaining angles and through 180° and/or 90°, so becomes Heron if all the remaining angles are chosen to be Heron at the outset.

Finally, we consider the Heron cyclic hexagon, in which three pairs of opposite sides are equal, so that if we denote the three distinct side lengths by a, b, c then the six sides are ordered in the sequence:

$$a, \quad b, \quad c, \quad a, \quad b, \quad c.$$

Furthermore, if α, β, γ are the three opposite peripheral angles then the relationship

$$\alpha + \beta + \gamma = 90°$$

must hold, where α, β, γ are Heron angles. Thus one has to begin with two arbitrary Heron angles α and β. Then one obtains:

$$a = 2r \sin \alpha, \quad b = 2r \sin \beta, \quad c = 2r \cos(\alpha + \beta).$$

Concerning the new diagonals, three of them are diameters and the remaining six are pairwise equal. If the three distinct lengths are denoted a', b', c' then we get:

$$a' = 2r \cos \alpha, \quad b' = 2r \cos \beta, \quad c' = 2r \sin(\alpha + \beta).$$

One obtains the area J of the Heron cyclic hexagon in numerous ways to be:

$$J = 4r^2 \cos \alpha \cos \beta \cos \gamma.$$

As an example we first choose $\sin \alpha = \frac{3}{5}$, $\sin \beta = \frac{5}{13}$, $\cos \alpha = \frac{4}{5}$, $\cos \beta = \frac{12}{13}$, $2r = 65$, and secondly $\sin \alpha = \frac{3}{5}$, $\sin \beta = \frac{8}{17}$, $\cos \alpha = \frac{4}{5}$, $\cos \beta = \frac{15}{17}$. Then it follows that:

1)	2)
$a = 39$	$a = 51$
$b = 25$	$b = 40$
$c = 33$	$c = 36$
$a' = 52$	$a' = 68$
$b' = 60$	$b' = 75$
$c' = 56$	$c' = 77$
$J = 2688$	$J = 4620$

9 Triangles with three rational sides and two rational medians

In §5 it was shown that only one of the three medians t_a, t_b, t_c of a Heron triangle can be rational. Therefore it is impossible to form triangles which have three rational sides a, b, c, rational area and more than one rational median. If one does without the rationality of the area this leads to soluble problems, namely those which require the rationality of the three sides and two of the medians. From the well known relationship:

$$(2t_a)^2 + a^2 = 2b^2 + 2c^2 = (b+c)^2 + (b-c)^2$$

which next leads to:

$$\left(\pm t_a + \frac{b-c}{2}\right)\left(\pm t_a - \frac{b-c}{2}\right) = s(s-a) \tag{9.1}$$

where s denotes the semi-perimeter $\frac{1}{2}(a+b+c)$. If a, b, c, t_a are rational then so are each of the factors on both sides of the equation (9.1). Therefore, from equation (9.1) it follows that one can equate two rational fractions and call the common value x, namely:

$$\frac{\pm t_a - \frac{1}{2}(b-c)}{s} = \frac{s-a}{\pm t_a + \frac{1}{2}(b-c)} = x. \tag{9.2}$$

From this it follows that:

$$\begin{cases} \pm\, t_a - \dfrac{1}{2}(b-c) = s \cdot x \\ \pm\, t_a + \dfrac{1}{2}(b-c) = \dfrac{s-a}{x} \end{cases} \tag{9.3}$$

or

$$\begin{cases} \pm\, 2t_a = (s-a)\left(x + \dfrac{1}{x}\right) + (s-b)x + (s-c)x \\ (s-b) - (s-c) = (s-a)\left(x - \dfrac{1}{x}\right) + (s-b)x + (s-c)x. \end{cases} \tag{9.4}$$

The second of the two equations (9.4) is a relationship between x and only the three sides. This can be rewritten as follows:

$$0 = \frac{s-a}{x} + \frac{s-b}{x+1} + \frac{s-c}{x-1}. \tag{9.5}$$

Any rational values of x, $s-a$, $s-b$, $s-c$ which satisfy this equation must imply that the median t_a is also rational.

For the triangle to be proper, i.e. the sum of any two sides is greater than the third, requires that $s-a$, $s-b$, $s-c$ be positive. This implies that the value of x lies between either 0 and $+1$ or between 0 and -1. Because otherwise in equation (9.5) either the sum of three positive values or the sum of three negative values would have to equal zero. Thus one always obtains a proper triangle when one satisfies equation (9.5) with a positive or negative proper fraction for x. To arrange for t_b to also be rational by symmetry we consider the analog to equation (9.5):

$$0 = \frac{s-b}{y} + \frac{s-c}{y+1} + \frac{s-a}{y-1}, \tag{9.6}$$

where now y also denotes a positive or negative fraction. Through elimination, firstly of $s-c$ and secondly of $s-b$, one obtains the system of equations:

$$\begin{cases} (s-a)\cdot\dfrac{2x+y-1}{x(x-1)(y-1)(y+1)} = (s-b)\cdot\dfrac{-x-2y-1}{(x-1)(x+1)y(y+1)} \\ (s-a)\cdot\dfrac{-x+y-1}{x(x+1)y(y-1)} = (s-c)\cdot\dfrac{-x-2y-1}{(x-1)(x+1)y(y+1)}. \end{cases} \tag{9.7}$$

Since x and y are positive or negative proper fractions, we substitute $1-x$, $1+x$, $1-y$, $1+y$ into the equations (9.7) and arrange it into the following proportion:

$$\frac{s-a}{(x+2y+1)x(1-y)} = \frac{s-b}{(2x+y-1)(1+x)y} = \frac{s-c}{(x-y+1)(1-x)(1+y)}. \tag{9.8}$$

Since we are only interested in the relationship between a, b, and c and thus between $s-a$, $s-b$, $s-c$, we can set:

$$\begin{cases} s-a = (x+2y+1)x(1-y) \\ s-b = (2x+y-1)(1+x)y \\ s-c = (x-y+1)(1-x)(1+y). \end{cases} \tag{9.9}$$

Through addition of the three equalities summarised in (9.9) one obtains:

$$s = 3xy + x - y + 1. \tag{9.10}$$

If one sets x and y to any possible positive or negative proper fractions, then one obtains from (9.9) and (9.10) rational values for s, $s-a$, $s-b$, $s-c$, from which follow rational values for a, b, c, and from these one finds that t_a and t_b must be rational, since (9.4) gives:

$$\begin{aligned} \pm 2t_a &= s \cdot x + \frac{s-a}{x} \\ \pm 2t_b &= s \cdot y + \frac{s-b}{y}. \end{aligned} \tag{9.11}$$

If one wants to arrange for the rational values a, b, c, t_a, t_b to be integers then one needs to still multiply by the common denominator. For example, from the following arbitrarily chosen proper fractions:

$$x = +\frac{1}{2}, \quad y = +\frac{1}{3}$$

we get:

$$\begin{aligned} s-a &= \left(\frac{1}{2} + 2 \cdot \frac{1}{3} + 1\right) \cdot \frac{1}{2} \cdot \frac{2}{3} = \frac{13}{18}, \\ s-b &= \left(2 \cdot \frac{1}{2} + \frac{1}{3} - 1\right) \cdot \frac{3}{2} \cdot \frac{1}{3} = \frac{3}{18}, \\ s-c &= \left(\frac{1}{2} - \frac{1}{3} + 1\right) \cdot \frac{1}{2} \cdot \frac{4}{3} = \frac{14}{18}, \\ s &= 3 \cdot \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} - \frac{1}{3} + 1 = \frac{30}{18}. \end{aligned}$$

Now we ignore the denominator 18, and so set $s-a = 13$, $s-b = 3$, $s-c = 14$, $s = 30$, from which we conclude:

$$a = s-(s-a) = \mathbf{17}, \quad b = s-(s-b) = \mathbf{27}, \quad c = s-(s-c) = \mathbf{16},$$

and find from (9.11) that to avoid the fraction $\frac{1}{2}$ we consider, not the ordinary but rather the double medians $2t_a$ and $2t_b$, namely:

$$2t_a = 41, \quad 2t_b = 19.$$

Actually we have:

$$41^2 + 17^2 = (27+16)^2 + (27-16)^2$$

and

$$19^2 + 27^2 = (17+16)^2 + (17-16)^2.$$

To show that negative proper fractions can also be used for x and y we associate to $x = \frac{1}{2}$ secondly $y = -\frac{1}{3}$. From this we obtain:

$$s - a = \frac{10}{18}, \quad s - b = \frac{3}{18}, \quad s - c = \frac{11}{18}, \quad s = \frac{24}{18},$$

hence

$$a = 14, \quad b = 21, \quad c = 13, \quad 2t_a = 32, \quad 2t_b = 17.$$

Actually, we have:

$$14^2 + 32^2 = (21 + 13)^2 + (21 - 13)^2$$

and:

$$21^2 + 17^2 = (14 + 13)^2 + (14 - 13)^2.$$

Since $s-a$, $s-b$, $s-c$ must be simultaneously positive or simultaneously negative then because of (9.9), if $y > x + 1$ and hence $s - c$ is negative, one considers whether $s - a$ and $s - b$ are also negative. However, this is always the case since the inequality $y > 1 + x$ implies that x must be negative and y must be positive. Here we present the third example:

$$x = -\frac{1}{2}, \quad y = +\frac{2}{3}.$$

Next one obtains:

$$s - a = -\frac{11}{36}, \quad s - b = -\frac{16}{36}, \quad s - c = -\frac{15}{36}, \quad s = -\frac{42}{36}.$$

We need to multiply by -36 to obtain positive integers and obtain:

$$s - a = 11, \quad s - b = 16, \quad s - c = 15, \quad s = 42,$$

$$a = 31, \quad b = 26, \quad c = 27,$$

$$2t_a = 43, \quad 2t_b = 52.$$

Actually we have:

$$43^2 + 31^2 = (27 + 26)^2 + (27 - 26)^2$$
$$52^2 + 26^2 = (31 + 27)^2 + (31 - 27)^2.$$

10 Triangles with three rational sides and three rational medians

Clearly, the path followed in §9 which led to triangles with three rational sides a, b, c and two rational medians t_a and t_b suggests that we can ask for triangles in which the three sides a, b, c and all three medians t_a, t_b, t_c are rational. One only needs to add a third condition to the two equations (9.5) (9.6), namely:

$$0 = \frac{s-c}{z} + \frac{s-a}{z+1} + \frac{s-b}{z-1},$$

where z is a positive or negative proper fraction, just the same as x and y from §9, so that the sought for triangle is non-degenerate. Thus our problem is equivalent to the the solution to the following system of equations:

$$\begin{cases} 0 = \dfrac{s-a}{x} + \dfrac{s-b}{x+1} + \dfrac{s-c}{x-1}, \\ 0 = \dfrac{s-b}{y} + \dfrac{s-c}{y+1} + \dfrac{s-a}{y-1}, \\ 0 = \dfrac{s-c}{z} + \dfrac{s-a}{z+1} + \dfrac{s-b}{z-1}, \end{cases} \tag{10.1}$$

in rational values for $s-a$, $s-b$, $s-c$.

For a solution to exist, the following condition between the fractions x, y, z must be satisfied:

$$0 = \begin{vmatrix} \frac{1}{x} & \frac{1}{1+x} & -\frac{1}{1-x} \\ -\frac{1}{1-y} & \frac{1}{y} & \frac{1}{1+y} \\ \frac{1}{1+z} & -\frac{1}{1-z} & \frac{1}{z} \end{vmatrix}, \tag{10.2}$$

or, on expansion of the determinant:

$$\begin{aligned} &\frac{1}{x}\cdot\frac{1}{y}\cdot\frac{1}{z} + \frac{1}{1+x}\cdot\frac{1}{1+y}\cdot\frac{1}{1+z} - \frac{1}{1-x}\cdot\frac{1}{1-y}\cdot\frac{1}{1-z} \\ &= \frac{1}{x}\cdot\frac{1}{1+y}\cdot\frac{1}{1-z} + \frac{1}{1+x}\cdot\frac{1}{1-y}\cdot\frac{1}{z} + \frac{1}{1-x}\cdot\frac{1}{y}\cdot\frac{1}{1+z}. \end{aligned} \tag{10.3}$$

Every triple of positive or negative proper fractions x, y, z which satisfy equation (10.3) must lead to a triangle with three rational sides and three rational medians. In order to compute a, b, c, t_a, t_b, t_c, from such a triple we use equations (9.9) and (9.10) which provide us with s, $s-a$, $s-b$, $s-c$ and hence a, b, c. Finally, one obtains the medians with equation (9.4). For example, the equation (10.3) is satisfied, if one sets $x = \frac{9}{13}$, $y = \frac{7}{13}$, $z = -\frac{25}{52}$. From this we get:

$$s-a = (x+2y+1)x(1-y) = \left(\frac{9}{13}+\frac{14}{13}+\frac{13}{13}\right)\cdot\frac{9}{13}\cdot\frac{6}{13} = \frac{36\cdot 9\cdot 6}{13^3},$$

$$s-b = (2x+y-1)(1+x)y = \left(\frac{18}{13}+\frac{7}{13}-\frac{13}{13}\right)\cdot\frac{22}{13}\cdot\frac{7}{13} = \frac{12\cdot 22\cdot 7}{13^3},$$

$$s-c = (x-y+1)(1-x)(1+y) = \left(\frac{9}{13}-\frac{7}{13}+\frac{13}{13}\right)\cdot\frac{4}{13}\cdot\frac{20}{13} = \frac{15\cdot 4\cdot 20}{13^3},$$

$$s = 3xy+x-y+1 = 3\cdot\frac{9}{13}\cdot\frac{7}{13}+\frac{9}{13}-\frac{7}{13}+\frac{13}{13} = \frac{384}{13^2}.$$

To obtain integers, we multiply the fractions obtained for $s-a$, $s-b$, $s-c$, s, by $\frac{13^3}{24}$ and obtain:

$$s = 208, \quad s-a = 81, \quad s-b = 77, \quad s-c = 50.$$

From this it follows that:

$$a = 127, \quad b = 131, \quad c = 158.$$

Then we obtain the three medians from:

$$\pm 2t_a = s\cdot x + \frac{s-a}{x} = 208\cdot\frac{9}{13} + 81\cdot\frac{13}{9} = 261,$$
$$\pm 2t_b = s\cdot y + \frac{s-b}{y} = 208\cdot\frac{7}{13} + 77\cdot\frac{13}{7} = 255,$$
$$\pm 2t_c = s\cdot z + \frac{s-c}{z} = -208\cdot\frac{25}{52} - 50\cdot\frac{52}{25} = -204.$$

So the three double-medians $2t_a = 261$, $2t_b = 255$, $2t_c = 204$ correspond to the three sides $a = 127$, $b = 131$, $c = 158$. Before we begin with a general solution of equation (10.3) into proper rational fractions, we first mention that each solution of the problem to produce three rational sides and three rational medians, which is easily recognisable, leads to a new solution in which one-third part of the medians become the sides of a new triangle whose double-medians are the sides of the old solution. For example, the above solution gives:

$$a = 127, \quad b = 131, \quad c = 158, \quad 2t_a = 261, \quad 2t_b = 255, \quad 2t_c = 204$$

which implies the new solution:

$$a = {}^1/_3 \cdot 261 = 87, \quad b = {}^1/_3 \cdot 255 = 85, \quad c = {}^1/_3 \cdot 204 = 68,$$

to which the double medians:

$$2t_a = 127, \quad 2t_b = 131, \quad 2t_c = 158,$$

correspond.

To provide a general method to obtain proper rational fractions, which when substituted for x, y, z satisfy equation (10.3), we order the equality by z after multiplication by $x(1-x)(1+x)y(1-y)(1+y)$ to obtain the equality:

$$\frac{1}{1+z} \cdot (x+2y+1)x(1-y) - \frac{1}{1-z} \cdot (2x+y-1)(1+x)y + \frac{1}{z}(x-y+1)(1-x)(1+y) = 0. \tag{10.4}$$

This equation is quadratic for each of the three unknowns. We now deal with a methodical method to find rational values for x, y, z that satisfy this equation. Note that x, y, z must be simultaneously positive or simultaneously negative proper fractions, while $s-a$, $s-b$, $s-c$, must be positive. All these conditions can be easily fulfilled by decomposing equation (10.4) into two linear equations in z and then paying attention to the following. Firstly, in the decomposition one needs to avoid the occurrence of all three of the three related fractions using the same unknown, namely

$$\frac{1}{z}, \quad \frac{1}{1-z}, \quad \frac{1}{1+z},$$

in either of the two sums which when set to zero and combined result in equation (10.4), since otherwise a quadratic equation would result for this unknown. Secondly, none of the three products multiplied by $\frac{1}{z}$, $\frac{1}{1-z}$, $\frac{1}{1+z}$, or one of the three analogous fractions, are allowed to be zero, otherwise expressions proportional to

$$s-a, \quad s-b, \quad s-c$$

would be zero. Thirdly, one needs to arrange that the two expressions, which were obtained from the two sums of one and the same unknown, z

again, that were both set to zero, are constructed in such a way that when equating the two expressions the resulting equality is linear in x or y. These requirements suffice to produce a whole series of decompositions, of which we emphasise nine, which lead to nine infinite series of triples of rational values of x, y, z and hence lead to nine series of triples of integers, which when substituted for the sides of a triangle cause the triangle to also have three integer double-medians. To clearly present these nine decompositions we denote the numerators arising in equation (10.4) corresponding to the denominators $1+z$, $1-z$, z by A, B, C and then it is understood that

$$A_1, \quad A_2; \quad B_1, \quad B_2; \quad C_1, \quad C_2$$

are expressions, which depend on x and y, with the property that

$$A_1 + A_2 = A, \quad B_1 + B_2 = B, \quad C_1 + C_2 = C.$$

Each of the three expressions

$$\begin{aligned} A &= (x+2y+1)x(1-y), \\ B &= (2x+y-1)(1+x)y, \\ C &= (x-y+1)(1-x)(1+y) \end{aligned}$$

contains three factors, of which two come from the expressions x, $1-x$, $1+x$, y, $1-y$, $1+y$, z, $1-z$, $1+z$, while the third factor is presented differently for each of the nine decompositions, namely:

$$\begin{aligned}
&\text{in 1):} & x+2y+1 &= [3y] + [x-y+1], \\
&\text{in 2):} & 2x+y-1 &= [-3(1+y)] + [2(x+2y+1)], \\
&\text{in 3):} & x-y+1 &= [{}^3/_2\,(1-y)] + [{}^1/_2\,(2x+y-1)], \\
&\text{in 4) and 7):} & x+2y+1 &= [(1+x)(1+y)] + [(1-x)y], \\
&\text{in 5) and 8):} & 2x+y-1 &= [x(1+y)] + [-(1-x)(1-y)], \\
&\text{in 6) and 9):} & x-y+1 &= [(1+x)(1-y)] + [xy].
\end{aligned}$$

Through these nine decompositions one obtains nine systems of pairs of equations which are linear in z and are constructed in such a way that upon elimination of z the result is a linear equation in either x or y which permits one to obtain one of these two latter unknowns rationally in terms of the other. The nine pairs of equations are as follows:

$$\text{I} \quad \begin{cases} \frac{3yx(1-y)}{1+z} - \frac{(2x+y-1)(1+x)y}{1-z} = 0 \\ \frac{(x-y+1)x(1-y)}{1+z} + \frac{(x-y+1)(1-x)(1+y)}{z} = 0 \end{cases}$$

$$\text{II} \quad \begin{cases} \frac{3(1+y)(1+x)y}{1-z} + \frac{(x-y+1)(1-x)(1+y)}{z} = 0 \\ -\frac{2(x+2y-1)(1+x)y}{1-z} + \frac{(x+2y+1)x(1-y)}{1+z} = 0 \end{cases}$$

$$\text{III} \quad \begin{cases} \frac{\frac{3}{2}(1-y)(1-x)(1+y)}{z} + \frac{(x+2y+1)x(1-y)}{1+z} = 0 \\ \frac{\frac{1}{2}(2x+y-1)(1-x)(1+y)}{z} - \frac{(2x+y-1)(1+x)y}{1-z} = 0 \end{cases}$$

$$\text{IV} \quad \begin{cases} \frac{(1+x)(1+y)x(1-y)}{1+z} + \frac{(x-y+1)(1-x)(1+y)}{z} = 0 \\ \frac{(1-x)yx(1-y)}{1+z} - \frac{(2x+y-1)(1+x)y}{1-z} = 0 \end{cases}$$

$$\text{V} \quad \begin{cases} -\frac{x(1+y)(1+x)y}{1-z} + \frac{(x-y+1)(1-x)(1+y)}{z} = 0 \\ \frac{(1-x)(1-y)(1+x)y}{1-z} + \frac{(x+2y+1)x(1-y)}{1+z} = 0 \end{cases}$$

$$\text{VI} \quad \begin{cases} \frac{(1+x)(1-y)(1-x)(1+y)}{z} + \frac{(x+2y+1)x(1-y)}{1+z} = 0 \\ \frac{xy(1-x)(1+y)}{z} - \frac{(2x+y-1)(1+x)y}{1-z} = 0 \end{cases}$$

$$\text{VII} \quad \begin{cases} \frac{(1+x)(1+y)x(1-y)}{1+z} - \frac{(2x+y-1)(1+x)y}{1-z} = 0 \\ \frac{(1-x)yx(1-y)}{1+z} + \frac{(x-y+1)(1-x)(1+y)}{z} = 0 \end{cases}$$

$$\text{VIII} \quad \begin{cases} -\frac{x(1+y)(1+x)y}{1-z} + \frac{(x+2y+1)x(1-y)}{1+z} = 0 \\ \frac{(1-x)(1-y)(1+x)y}{1-z} + \frac{(x-y+1)(1-x)(1+y)}{z} = 0 \end{cases}$$

$$\text{IX} \quad \begin{cases} \frac{(1+x)(1-y)(1-x)(1+y)}{z} - \frac{(2x+y-1)(1+x)y}{1-z} = 0 \\ \frac{xy(1-x)(1+y)}{z} + \frac{(x+2y+1)x(1-y)}{1+z} = 0. \end{cases}$$

In each of the 18 equations, summarised as 9 pairs of equations, one is allowed to remove a factor of

$$x, \quad 1-x, \quad 1+x, \quad y, \quad 1-y, \quad 1+y$$

or

$$x+2y+1, \quad 2x+y-1, \quad x-y+1$$

since none of these factors are allowed to be zero. Clearly, if one of the first six factors was equal to zero then either x or y would have to equal 0, +1, or −1, which is impossible. And if one of the last three factors was equal to zero then one of $s-a$, $s-b$ or $s-c$ must be zero, which is impossible. Because of this removal one of the factors one obtains, on elimination of z from each of the nine systems, an equation in x and y which is linear in y for I, II, III, IV, V, VI and is linear in x for VII, VIII, IX. So one obtains a rational expression for y in terms of x in the first six cases and a rational expression for x in terms of y in the last three cases. Since z is bound to

x and y via two equalities which are linear in z, in each of the nine cases, this solves the problem of finding an infinite number of rational triples for x, y, z. But we still need to discuss how one ensures that all three values are positive or negative proper fractions, because only by this constraint is it possible to obtain proper triangles. The rational functions obtained from the nine systems of equations are:

$$y = \frac{7-4x-2x^2}{-5+10x}; \tag{10.5}$$

$$y = \frac{4+3x-4x^2}{10+5x}; \tag{10.6}$$

$$y = \frac{-3+2x-x^2}{6+2x}; \tag{10.7}$$

$$y = \frac{(1+x)(3-4x-x^2)}{(3-x)(1+x^2)}; \tag{10.8}$$

$$y = \frac{(1+x)(-2+4x-3x^2)}{(2-x)(-1+2x+x^2)}; \tag{10.9}$$

$$y = \frac{(1+x)(1-3x-x^2)}{(1+2x)(1+2x-x^2)}; \tag{10.10}$$

$$x = \frac{(1-y)(2+4y+3y^2)}{(2+y)(-1-2y+y^2)}; \tag{10.11}$$

$$x = \frac{(1-y)(-3-4y+y^2)}{(3+y)(1+y^2)}; \tag{10.12}$$

$$x = \frac{(1-y)(-1-3y+y^2)}{(1-2y)(1-2y-y^2)}. \tag{10.13}$$

The nine formulæ above which lead to rational solutions of equation (10.4), are not unrelated. One observes that (10.11) is obtained from (10.9) if one replaces x by $-y$ and y by $-x$ and in a similar way (10.12) is related to (10.8) and (10.13) to (10.10). Thus we completely ignore the decomposition of equations (10.11), (10.12), (10.13). Similarly the decomposition of equations (10.8), (10.9), (10.10) are related to the decomposition of equations (10.5), (10.6), (10.7). One recognises this, in the six decompositions, if one expresses the eliminated rational value z first in terms of x and y and then just in terms of x.

How the values:

$$s-a, \quad s-b, \quad s-c, \quad s, \quad a, \quad b, \quad c, \quad 2t_a, \quad 2t_b, \quad 2t_c$$

are obtained from each of the triples of proper fractions x, y, z has already been shown above in the example:

$$x = \frac{9}{13}, \quad y = \frac{7}{13}, \quad z = -\frac{25}{52}.$$

One discovers how (10.8), (10.9), (10.10) are related to the formulæ (10.7), (10.6), (10.5) by expressing z in terms of x for (10.5), (10.6), (10.7). One obtains:

$$z = -\frac{(1-x)(1+3x-x^2)}{(1-2x)(1-2x-x^2)}, \tag{10.14}$$

$$z = \frac{(1-x)(2+4x+3x^2)}{(2+x)(-1-2x+x^2)}, \tag{10.15}$$

$$z = -\frac{(1-x)(3+4x-x^2)}{(3+x)(1+x^2)}. \tag{10.16}$$

One observes that (10.8) is derived from (10.16), (10.9) from (10.15), (10.10) from (10.14) if one replaces x by $-x$ and y by $-z$.

We already observed that during application of the above system of equalities one must ensure that y and z are proper rational fractions once a positive or negative proper fraction was chosen for x. This question leads naturally to a discussion of, for example, the second of the nine cases. In this case:

$$y = \frac{4+3x-4x^2}{10+5x}.$$

To ensure that y is also a proper fraction, once x has been set equal to a proper fraction, we must have

$$-10-5x < 4+3x-4x^2 < 10+5x$$

or

$$(1-x)^2 < \frac{9}{2} \quad \text{and} \quad \left(\frac{1}{4}+x\right) > -\frac{23}{16}.$$

These inequalities are always satisfied when x lies between -1 and $+1$. Thus in the second of the nine cases, for each proper fraction x there is also a value for y which lies between -1 and $+1$. If e.g. $x = +\frac{2}{3}$, one obtains $y = \frac{19}{60}$, from which $s-a$, $s-b$, $s-c$, s, a, b, c, $2t_a$, $2t_b$, $2t_c$ follow and hence z must be a proper fraction, namely $z = -\frac{27}{68}$.

We now consider an example for each of the six paths by which x can be chosen so that y is also a proper positive or negative fraction. The numbers correspond to those used above for the decomposition of the equation (10.4).

$$\text{I}\quad \begin{aligned} &x=+\frac{4}{5},\quad y=+\frac{21}{25},\quad z=-\frac{23}{31},\\ &s-a=116,\quad s-b=567,\quad s-c=92,\quad s=775,\\ &a=659,\quad b=208,\quad c=683,\\ &2t_a=765,\quad 2t_b=1326,\quad 2t_c=699; \end{aligned}$$

$$\text{II}\quad \begin{aligned} &x=+\frac{2}{3},\quad y=+\frac{19}{60},\quad z=-\frac{27}{68},\\ &s-a=3772,\quad s-b=1235,\quad s-c=2133,\quad s=7140,\\ &a=3368,\quad b=5905,\quad c=5007,\\ &2t_a=10418,\quad 2t_b=6161,\quad 2t_c=8207; \end{aligned}$$

$$\text{III}\quad \begin{aligned} &x=+\frac{1}{2},\quad y=-\frac{9}{28},\quad z=-\frac{19}{35},\\ &s-a=296,\quad s-b=81,\quad s-c=323,\quad s=700,\\ &a=404,\quad b=619,\quad c=377,\\ &2t_a=942,\quad 2t_b=477,\quad 2t_c=975; \end{aligned}$$

$$\text{IV}\quad \begin{aligned} &x=+\frac{1}{2},\quad y=+\frac{9}{25},\quad z=-\frac{19}{35},\\ &s-a=296,\quad s-b=81,\quad s-c=323,\quad s=700,\\ &a=404,\quad b=619,\quad c=377,\\ &2t_a=942,\quad 2t_b=477,\quad 2t_c=975; \end{aligned}$$

$$\text{V}\quad \begin{aligned} &x=-\frac{2}{3},\quad y=+\frac{27}{68},\quad z=-\frac{65}{119},\\ &s-a=3772,\quad s-b=2133,\quad s-c=1235,\quad s=7140,\\ &a=3368,\quad b=5007,\quad c=5905,\\ &2t_a=10418,\quad 2t_b=8207,\quad 2t_c=6161; \end{aligned}$$

$$\text{VI}\quad \begin{aligned} &x=\frac{2}{5},\quad y=-\frac{7}{41},\quad z=-\frac{51}{82},\\ &s-a=744,\quad s-b=133,\quad s-c=1173,\quad s=2050,\\ &a=1306,\quad b=1917,\quad c=877,\\ &2t_a=2680,\quad 2t_b=1129,\quad 2t_c=3161. \end{aligned}$$

The examples III and IV as well as II and V show that one can obtain the same six integers for a, b, c, $2t_a$, $2t_b$, $2t_c$ even when one begins with two different rational triples for x, y, z. In the two identical cases, III and IV, as well as case I, one can use the one-thirding procedure, mentioned above, to generate a new solution with smaller numbers. Recall that this procedure leads to a solution in which one third of each of the double-medians become the new sides of a triangle whose double-medians are the sides of the original triangle. From this it follows that I gives:

$$a = 255, \quad b = 442, \quad c = 233,$$
$$2t_a = 659, \quad 2t_b = 208, \quad 2t_c = 683.$$

Likewise, it follows from III or IV that:

$$a = 314, \quad b = 159, \quad c = 325,$$
$$2t_a = 404, \quad 2t_b = 619, \quad 2t_c = 377.$$

In order to have an example for the three last decompositions (10.11), (10.12), (10.13), we set $y = +\frac{1}{3}$, in (10.11), from which we obtain: $x = -\frac{33}{49}$ and $z = -\frac{2}{35}$. This leads to:

$$s - a = 2409, \quad s - b = 1184, \quad s - c = 82, \quad s = 3675,$$
$$a = 1266, \quad b = 2491, \quad c = 3593,$$
$$2t_a = 6052, \quad 2t_b = 4777, \quad 2t_c = 1645.$$

Shortly before the author, Freiherr von Thielmann (in the village of Kreuth, Oberbayern, now returned to Berlin) followed a completely different path to produce a method which also allowed him to obtain an infinite number of triangles whose sides and double-medians are simultaneously integral. Freiherr von Thielmann has kindly permitted the publication of his 7 examples here:

(1) $a = 87, b = 85, c = 68; 2t_a = 127, 2t_b = 131, 2t_c = 158;$

(2) $a = 619, b = 377, c = 404; 2t_a = 477, 2t_b = 975, 2t_c = 942;$

(3) $a = 446, b = 277, c = 477; 2t_a = 640, 2t_b = 881, 2t_c = 569;$

(4) $a = 509, b = 1323, c = 1664; 2t_a = 2963, 2t_b = 2075, 2t_c = 1118;$

(5) $a = 13093, b = 8675, c = 17128; 2t_a = 23787, 2t_b = 29229, 2t_c = 14142;$

(6) $a = 4607, b = 3238, c = 6715; 2t_a = 9483, 2t_b = 11052, 2t_c = 4281;$

(7) $a = 26184, b = 19651, c = 15805; 2t_a = 24214, 2t_b = 38531, 2t_c = 43517.$

Several of the solutions from those of v. Thielmann are identical to our examples, but not all. Also there are some solutions above, found by the author, which do not occur in the examples just presented by v. Thielmann. Finally, we present those four examples in which none of the six numbers have more than three digits and where we always only show one of the two examples which are connected by the one-thirding procedure:

(1) $a = 87, b = 85, c = 68; 2t_a = 127, 2t_b = 131, 2t_c = 158;$

(2) $a = 159, b = 325, c = 314; 2t_a = 619, 2t_b = 377, 2t_c = 404;$

(3) $a = 446, b = 277, c = 477; 2t_a = 640, 2t_b = 881, 2t_c = 569;$

(4) $a = 255, b = 442, c = 233; 2t_a = 659, 2t_b = 208, 2t_c = 683.$

11 Heron pyramids

Above every Heron triangle (§2), quadrilateral (§7) or polygon (§8) one can construct innumerably many pyramids with rational sides and rational volume. Then it is easy to show that such pyramids, which we call Heron, also have a rational circum-sphere radius. Namely, if one erects a perpendicular to the plane of a cyclic polygon, at its circumcentre, and one connects an arbitrary point on the perpendicular with the sides of the polygon, then this creates a right pyramid in which all equal slant-lengths together with the height h and the radius r produce congruent triangles. Accordingly, if one arranges that the angle μ lying opposite h, in these congruent triangles, is Heron then the slant-lengths k and the height h must be rational with respect to the radius r. Hence the volume V of the pyramid must also be rational since the area of the cyclic polygon is rational. The radius R of the circumsphere, which lies on the surface of the circumcircle of the cyclic polygon and the apex of the pyramid, must also be rational since R can be rationally expressed in terms of r, $\sin\mu$, $\cos\mu$. One has:

$$k = \frac{r}{\cos\mu}; \quad h = r \cdot \tan\mu; \tag{11.1}$$

$$R = \frac{k^2}{2h} = \frac{r}{2\sin\mu\cos\mu} = \frac{r}{\sin 2\mu}. \tag{11.2}$$

One only has to ensure that the base of the pyramid is a Heron polygon (§2, §7, §8), and that the angle μ is Heron, so that the corresponding pyramid has rational sides, volume and radius R of the circumsphere. In connection with §2, §7, §8 and the above formulæ (11.1) and (11.2), one can naturally express the sides k, the volume V and the radius R in terms of the integers m, n, p, q as well as v, w where $\frac{w}{v}$ is the constituent of the Heron angle μ. Hence we obtain rational values for k, V, R whenever we set $m, n, p, q, \ldots, v, w$ to integer values. For brevity, we would like to consider just a few examples from §2, §7, §8 and connect them with arbitrarily chosen Heron angles μ, in accordance with formulæ (11.1) and (11.2). One can always multiply the resulting rational numbers by their common denominator to finally obtain integers:

1. In the first example of the list of all Heron triangles in §2, where we had $a = 13$, $b = 15$, $c = 14$, $J = 84$, we choose μ as the Heron angle with a cosine of $\frac{65}{97}$ and a sine of $\frac{72}{97}$. Then $r = \frac{65}{8}$ because $a = 13$, $b = 15$, $c = 14$, and we obtain:

$$k = \frac{65 \cdot 97}{8 \cdot 65} = \frac{97}{8}, \quad h = \frac{65 \cdot 72}{8 \cdot 65} = 9,$$
$$R = \frac{65 \cdot 97 \cdot 97}{8 \cdot 2 \cdot 72 \cdot 65} = \frac{97^2}{1152}, \quad V = \frac{J \cdot h}{3} = \frac{84 \cdot 9}{3} = 252.$$

If the six edges of the pyramid as well as the volume are required to be integers then, in accordance with these results, one needs to set

$$a = 104, \quad b = 120, \quad c = 112, \quad k = 97, \quad V = 129024,$$

where a, b, c are the sides of the base, k the slant sides of the pyramid, and V its volume.

2. In example 13 of the list in §2 we had $a = 85$, $b = 105$, $c = 76$, $J = 3192$, so that we could set $r = \frac{425}{8}$. If one associates to this the Heron angle μ with sine $\frac{132}{157}$ and cosine $\frac{85}{157}$, then one obtains:

$$k = \frac{425 \cdot 157}{8 \cdot 85} = \frac{785}{8}, \quad h = \frac{425 \cdot 132}{8 \cdot 85} = \frac{165}{2},$$
$$R = \frac{425 \cdot 157^2}{8 \cdot 2 \cdot 132 \cdot 85} = \frac{157^2}{16 \cdot 132}, \quad V = \frac{3192 \cdot 165}{2 \cdot 3} = 87780.$$

From this one obtains a triangular pyramid, with base edges of 680, 840, 608 and a slant edge of 785, whose volume equals 44943360.

3. In example (7) of the list of Heron cyclic quadrilaterals of §7 we had $a_1 = 85$, $a_2 = 50$, $a_3 = 102$, $a_4 = 45$, thus the radius r of the circumcircle is

$\frac{425}{8}$. We connect these results with $\cos\mu = \frac{425}{457}$, $\sin\mu = \frac{168}{457}$. Then we get:

$$k = \frac{425 \cdot 457}{8 \cdot 425} = \frac{457}{8}, \quad h = \frac{425 \cdot 168}{8 \cdot 425} = 21,$$
$$R = \frac{425 \cdot 457^2}{8 \cdot 2 \cdot 168 \cdot 425} = \frac{457^2}{16 \cdot 168}, \quad V = \frac{4368 \cdot 21}{2 \cdot 3} = 30576.$$

When the four base edges of a quadrilateral pyramid are 680, 400, 816, 360 and the slant edge is 457 then the volume equals 15654912.

4. We associate the example in §8 for the general Heron pentagon with the Heron angle whose cosine is $\frac{221}{229}$ and whose sine is $\frac{60}{229}$. Hence we obtain:

$$a_1 = 3315, \quad a_2 = 2125, \quad a_3 = 2600, \quad a_4 = 1547, \quad a_5 = 5365,$$
$$r = \frac{5525}{2}, \quad k = \frac{5525 \cdot 229}{2 \cdot 221} = \frac{25 \cdot 229}{2},$$
$$h = \frac{5525 \cdot 60}{2 \cdot 221} = 750, \quad R = \frac{5525 \cdot 229^2}{2 \cdot 2 \cdot 221 \cdot 60} = \frac{5 \cdot 229^2}{48},$$
$$V = \frac{13362972 \cdot 750}{3} = 3340743000.$$

Thus when the five base edges of a pentagonal pyramid are set to the five numbers 6630, 4250, 5200, 3094, 10730 and the five slant edges are set to the value 5625, then the volume is equal to

$$26725944000.$$

5. In the first of the two examples of cyclic hexagons at the end of §8 opposite pairs of sides were equal and were set to the values 39, 25, 33. For the area we obtained 2688, and for r we get $\frac{65}{2}$. We connect this result with:

$$\sin\mu = \frac{72}{97}, \quad \cos\mu = \frac{65}{97}$$

which leads to

$$k = \frac{65 \cdot 97}{2 \cdot 65} = \frac{97}{2}, \quad h = \frac{65 \cdot 72}{2 \cdot 65} = 36,$$
$$R = \frac{65 \cdot 97^2}{2 \cdot 2 \cdot 72 \cdot 65} = \frac{97^2}{288}, \quad V = \frac{2688 \cdot 36}{3} = 32256.$$

So if one sets the six base edges of a hexagonal right pyramid sequentially equal to:

$$78, 50, 66, 78, 50, 66$$

and one sets all six slant edges equal to 97 then the volume is also integral and equal to:

$$258048.$$

12 Square pyramids, whose eight edges and volume are integral

If a denotes the side of the square that forms the base of a right pyramid with height h then the volume V is:

$$V = \frac{1}{3} \cdot a^2 \cdot h = \frac{1}{3} \cdot a^2 \cdot \sqrt{b^2 - \frac{1}{2}a^2},$$

where b is the slant length. Now if in the equation:

$$b^2 - h^2 = \frac{1}{2} \cdot a^2$$

a, b, h are supposed to be integral, then a must be even hence a is divisible by four so that $b^2 - h^2$ is even. If a and b and hence also h have no common divisors then b and h must both be odd, thus:

$$\frac{b+h}{2} \cdot \frac{b-h}{2} = \frac{1}{2} \cdot \left(\frac{a}{2}\right)^2 = 2 \cdot \left(\frac{a}{4}\right)^2$$

where $\frac{b+h}{2}$, $\frac{b-h}{2}$, $\frac{a}{2}$ are integers which have no common factor. Since the product of the two integers $\frac{b+h}{2}$ and $\frac{b-h}{2}$ equals twice a square $\left(\frac{a}{2}\right)^2$ then one of the two integers must be a square v^2, while the other must equal twice a square, say $2w^2$. Thus we distinguish two cases, where we set either:

$$\begin{cases} b = v^2 + 2w^2 \\ h = v^2 - 2w^2 \\ a = 4vw \\ V = \dfrac{16v^2w^2}{3}(v^2 - 2w^2) \end{cases} \quad \text{or} \quad \begin{cases} b = 2v^2 + w^2 \\ h = 2v^2 - w^2 \\ a = 4vw \\ V = \dfrac{16v^2w^2}{3}(2v^2 - w^2). \end{cases}$$

To ensure that a and b are relatively prime we must set v odd in the first case and w odd in the second case. Similarly, V is integral if either v or w is divisible by 3. Now if neither v nor w were divisible by three, rather of the form $3n \pm 1$, then $v^2 - 2w^2$ would be of the form $3m - 1$ in the first case, and would be of the form $3m + 1$ in the second case so that V would be a fraction with a denominator of three. So that the four base edges a, the four slant edges b and the volume V are positive integers without common divisors, one needs to ensure that the following conditions are satisfied when choosing the integers v and w:

1. In the first case v must be odd and larger than $w\sqrt{2}$; in the second case w must be odd and smaller than $v\sqrt{2}$;

2. v and w must be relatively prime;

3. One of the two integers v and w must be divisible by three.

If one satisfies these conditions when systematically choosing v and w from small values of v to larger values then one obtains all possible solutions to the problem of constructing a square pyramid with eight integer edges and integer volume. The following list contains all resulting solutions for which v is no larger than ten. When two solutions are obtained from a single pair of integers, one from the first case the other from the second case, then both solutions are shown in the table.

13 Right pyramids, with a regular hexagon as base, twelve integer edges and integer volume

If a is the side of a regular hexagon then its area is $\frac{3}{2}a^2\sqrt{3}$. Therefore if h denotes its height, then its volume is:

$$V = \frac{1}{2}a^2h\sqrt{3},$$

or, if b denotes the edge length then:

$$V = \frac{1}{2}a^2\sqrt{3(b^2 - a^2)}.$$

For V to be rational, we must have $b^2 - a^2$ equal to three times a square. Therefore we set:

$$b^2 - a^2 = 3d^2.$$

If d is odd then b and a can neither be simultaneously even nor simultaneously odd. If b was odd and a was even then on division by four $b^2 - a^2$ would have a remainder of one, while that which it equals, namely $3d^2$,

	v	w	b	h	a	V
1)	3	1	11	7	12	336
2)	3	1	19	17	12	816
3)	3	2	17	1	24	192
4)	4	3	41	23	48	17664
5)	5	3	43	7	60	8400
6)	5	3	59	41	60	49200
7)	6	1	73	71	24	13632
8)	6	5	97	47	120	225600
9)	6	7	121	23	168	216384
10)	7	3	67	31	84	72912
11)	7	3	107	89	84	209328
12)	8	3	137	119	96	365568
13)	8	9	209	47	288	1299456
14)	9	1	83	79	36	34128
15)	9	1	163	161	36	69552
16)	9	2	89	73	72	126144
17)	9	4	113	49	144	338688
18)	9	5	131	31	180	334800
19)	9	5	187	137	180	1479600
20)	10	3	209	191	120	916800
21)	10	9	281	119	360	5140800
..	..	..	..	..	..	..

Table 8: Square pyramids whose eight edges and volume are integral.

would leave a remainder of three on division by four. Thus all that remains is for b to be even and a to be odd. Then we would have

$$V = \frac{1}{2}a^2 \cdot 3 \cdot d$$

which would be half an odd integer, thus rational but non-integral. Hence for a, b, V to be integers requires that d not be an odd integer. Therefore, each side of the equality

$$b^2 - a^2 = 3 \cdot d^2$$

is divisible by four, from which, since a and b are without common factors hence not both even, we can subsequently set

$$\frac{b+a}{2} \cdot \frac{b-a}{2} = 3 \cdot \left(\frac{d}{2}\right)^2,$$

that is the product of two integers equal to three times a square. From this follows that since the two integers

$$\frac{b+a}{2} \quad \text{and} \quad \frac{b-a}{2}$$

have no common factors, one must be a square and the other must be three times a square. From

$$\frac{b+a}{2} = v^2 \quad \text{and} \quad \frac{b-a}{2} = 3w^2$$

follows however:

$$b = v^2 + 3w^2; \quad a = v^2 - 3w^2.$$

Similarly, from

$$\frac{b+a}{2} = 3v^2 \quad \text{and} \quad \frac{b-a}{2} = w^2$$

it follows that

$$b = 3v^2 + w^2; \quad a = 3v^2 - w^2.$$

Hence we obtain for V either:

$$V = 3(v^2 - 3w^2)^2 vw,$$

or

$$V = 3(3v^2 - w^2)^2 vw.$$

If one sets v and w to arbitrary integers then one also obtains integers for a, b, V. To prevent common factors occurring in a and b we need to choose v and w in such a way that one is even and the other is odd, that in the first case v is not divisible by three and in the second case w is not divisible by three. So that a does not become negative we must ensure that $v > w\sqrt{3}$ in the first case and $w < v\sqrt{3}$ in the second case. If one implements these constraints then one always obtains integer values for a, b, V for which a and b are relatively prime, and actually a systematic increase from small integers for v and w to larger must lead to all such solutions to our problem. The following table shows how the formulæ above lead to integer values for a, b, V from the choice of small values of v and w:

	v	w	b	a	V
1)	2	1	7	1	6
2)	2	1	13	11	726
3)	3	2	31	23	9522
4)	3	4	43	11	4356
5)	4	1	19	13	2028
6)	4	1	49	47	26508
7)	4	5	73	23	31740
8)	5	2	37	13	5070
9)	5	2	79	71	151230
10)	5	4	91	59	208860
11)	5	8	139	11	14520
12)	6	1	109	107	206082
13)	6	5	133	83	620010
14)	6	7	157	59	438606
..	..	..	..	..	..

Table 9: Right pyramids with a regular hexagonal base, whose twelve edges and volume are integral.