Cyclic polygons with rational sides and area

Ralph Heiner Buchholz

November 4, 2005

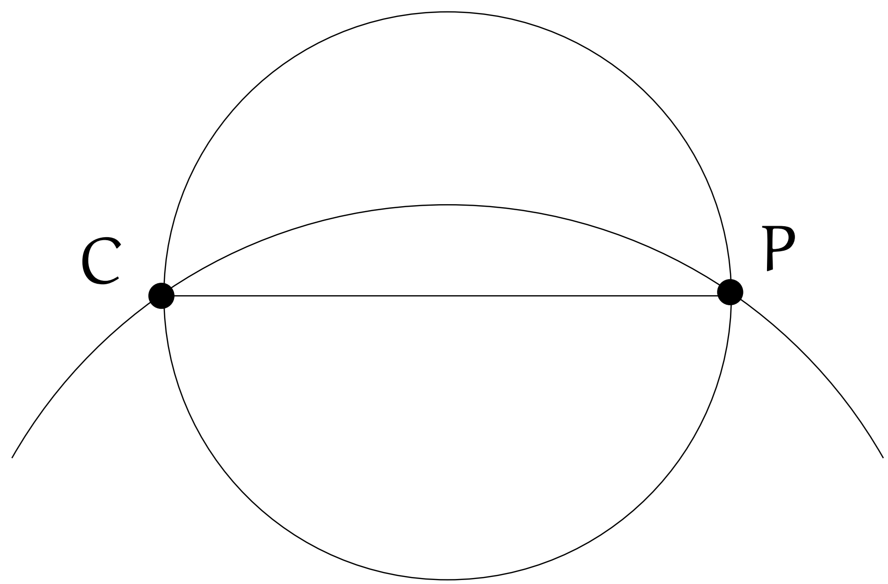
13,700,000,000 years ago

The big bang!

.

230,000 years ago

Homo Neanderthalis discovers that a straight line is the shortest distance from the cave, C, to the mammoth pit, P.



1650 B.C.

A^{c}h-mosè copies a crumbling papyrus about 200 years old.

250 B.C.

Archimedes (287 B.C. - 212 B.C.) steals "Heron's formula"

$$A_3 = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{a+b+c}{2}$. Meanwhile the circumradius is

$$R_3 = \frac{abc}{4A_3}.$$

650 A.D.

Brahmagupta (598 A.D. - 668 A.D.) discovers the formula for the area of a cyclic quadrilateral,

$$A_4 = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where $s = \frac{a+b+c+d}{2}$.

1430 A.D.

Parameśvara (1370 A.D. - 1460 A.D.) invents the circumradius formula for a cyclic quadrilateral

$$R_4 = \frac{\sqrt{(ac+bd)(ad+bc)(ab+cd)}}{4A_4}$$

1994 A.D.

Robbins (1942 A.D. - 2003 A.D.) revealed both the cyclic pentagon and hexagon formulæ.

$$2^{28}A_5^{14} + 2^{24}p_4A_5^{12} + \ldots + 2^4p_{24}A_5^2 + p_{28} = 0$$

$$2^{28}A_6^{14} + 2^{24}q_4A_6^{12} + \ldots + 2^4q_{24}A_6^2 + q_{28} = 0$$

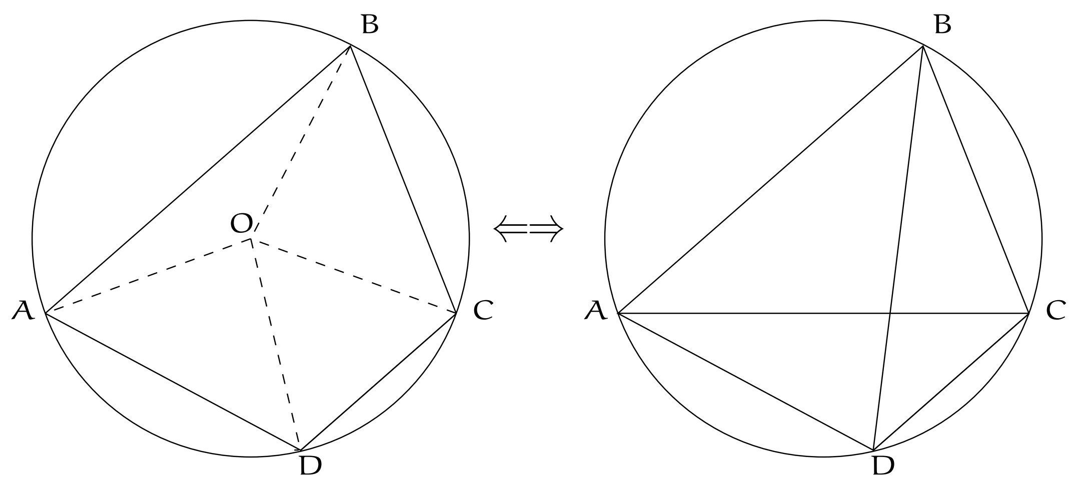
Heron triangles

- Euler (1781) — complete parametrization,
- Carmichael (1952) — more economical version,
- Buchholz (1989) — one member of each similarity class.

$$(a, b, c) = (n(m^2 + k^2), m(n^2 + k^2), (m + n)(mn - k^2))$$

$$\gcd(m, n, k) = 1, \quad m \geq n \geq 1, \quad mn > k^2 \geq \frac{m^2 n}{2m + n}.$$

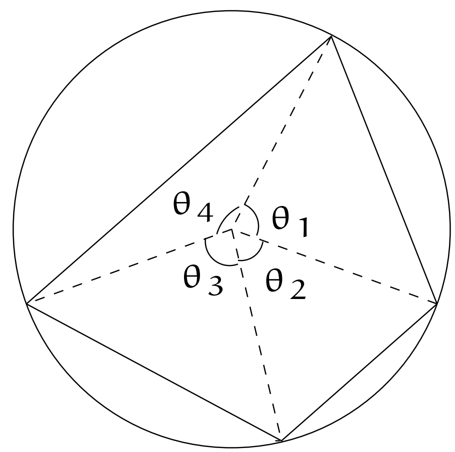
Cyclic rational area quadrilaterals



Radial versus diagonal decomposition

Cyclic rational area quadrilaterals

Euler defines: $\sin\theta_i = \frac{2p_i}{p_i^2+1}$ and $\cos\theta_i = \frac{p_i^2-1}{p_i^2+1}$ for $i = 1, 2, 3$,



so that $\theta_4 = \pi - \sum_{i=1}^{3}\theta_i$ is forced to be rational.

Cyclic rational area quadrilaterals

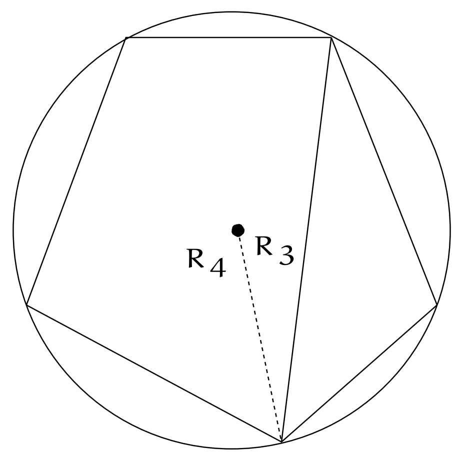
Buchholz and MacDougall show that the only possible circumradii include,

$$1, \sqrt{2}, \sqrt{5}, \sqrt{10}, \sqrt{13}, \sqrt{17}, \ldots, \sqrt{m}, \ldots$$

where $m = u^2 + v^2$ for integers u and v—in particular, no Brahmagupta quadrilateral can have a circumradius in $\sqrt{3}\mathbb{Q}$.

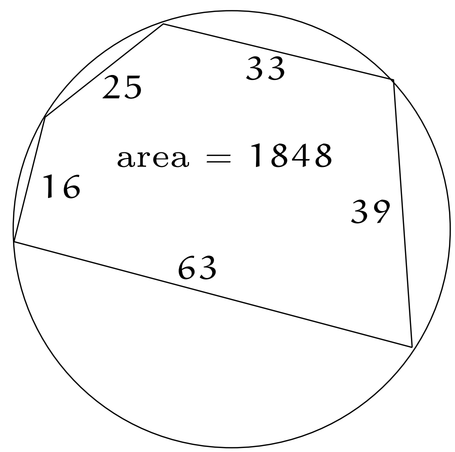
Cyclic rational area pentagons

$$\begin{bmatrix}\text{Robbins}\\ \text{pentagon}\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}\text{indecomposible}\\ \text{Brahmagupta}\\ \text{quadrilateral}\end{bmatrix} + \begin{bmatrix}\text{Heron}\\ \text{triangle}\end{bmatrix}$$



Cyclic rational area pentagons

- in theory — 0 or 5 rational diagonals
- in practice — all examples have 5 rational diagonals



Cyclic rational area pentagons

- Buchholz and MacDougall — diagonals satisfy a degree 7 polynomial in the sides
- Brown and Buchholz — diagonals satisfy a degree 4 polynomial in the sides.

Conjectures

- all odd-sided n-gons are radially decomposible
- all even-sided n-gons are either radially decomposible or quadrilaterally decomposible

Google

Ralph the triangle

I'm feeling lucky