An Epitrochoidal Mixer

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Abstract

This paper discusses a problem encountered in reducing the size of laboratory dough mixers. An approximate solution is initially derived to gain some understanding of the solution. Its solution is found by applying an iterative technique to a pair of non-linear equations.

Keywords : **Dough Mixer, Newtons Method, Simultaneous Equations.**

1 Introduction

Large batches of rare varieties of wheat (or other cereals) are sometimes difficult to obtain owing to low yields or small plots. Consequently, the testing of these varieties can be hampered because of the amounts required for even the smallest currently available mixers (about 30 cm bowl diameter). Clearly a smaller dough mixer would be an advantage - more test samples could be produced from a given volume of wheat. However, just scaling down the dimensions of an existing mixer causes several problems. The first problem is that the beater pins (see Figure 1) which move through the dough become so thin that they break. This occurs because the viscosity of the dough in the small mixer is still the same while the strength of each pin is reduced, as it is proportional to its cross sectional area. This can be solved by simply increasing the diameter of the pins until they no longer break during mixing. If this beater is used to mix

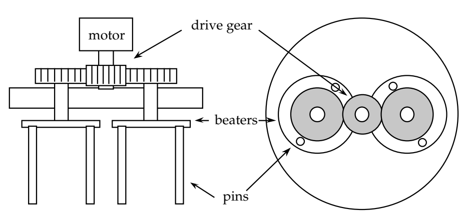

Figure 1: The Beater

dough, the contents of a stationary bowl would just slurry around with the beater pins. To mimic the kneading and folding action of a baker more closely, pins are required in the bowl as well as on the beater. These bowl pins restrict the movement of the dough, and so assist in proper mixing. The strain on the dough varies with time during the mixing and must conform qualitatively to a standard graph to ensure consistent repeatable dough qualities. The second problem which then arises is to find the position of the pins (in this case three) to be placed in a stationary bowl so that they are completely missed by the rotary motion of the four beater pins.

This problem was first communicated to the author by Chris Rath of the Bread Division, CSIRO. He was specifically involved in constructing a miniature dough mixer (3 cm bowl diameter) using large existing mixers as a model. The relative increase in pin diameter made empirical placement of the bowl pins rather difficult.

2 The Mathematical Model

First consider the path followed by a single pin on one of the beaters and then add the other pins as new paths. The position of a point on the circumference of the small circle in Figure 2 is given by

$$x = r_1 \cos\theta + r_2 \cos\phi$$
$$y = r_1 \sin\theta + r_2 \sin\phi.$$

The gear ratio,

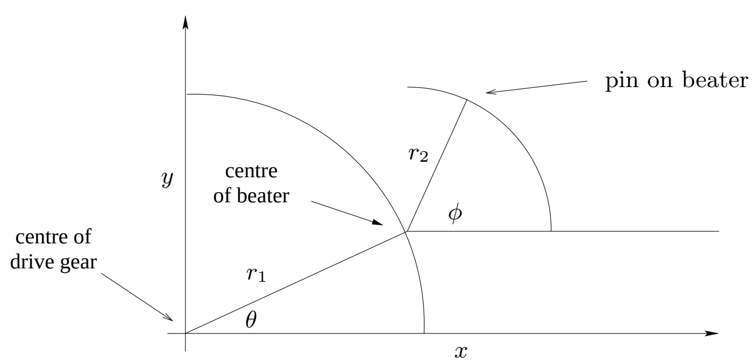

Figure 2: Geometry of the beater pin

$$\frac{n}{m} := \frac{\text{No. of teeth on drive gear}}{\text{No. of teeth on beater gear}}$$

gives us the relationship between the two angles θ and ϕ . When r_1 sweeps out one revolution about the origin the centre of the beater sweeps out one revolution about the drive gear, while r_2 sweeps out n/m revolutions about the centre of the beater. So r_2 sweeps out $(1 + n/m)$ revolutions about the origin i.e. $\phi = (1 + n/m)\theta$.

This means that the path of one pin is defined by

$$x = r_1 \cos(\theta) + r_2 \cos\left(\left(1 + \frac{n}{m}\right)\theta\right)$$
$$y = r_1 \sin(\theta) + r_2 \sin\left(\left(1 + \frac{n}{m}\right)\theta\right),$$

or in polar coordinates an *epitrochoid*

$$r^2 = r_1^2 + r_2^2 + 2r_1 r_2 \cos\left(\frac{n}{m}\theta\right), \tag{1}$$

where $r_1 = 7.1755$mm, $r_2 = 5.207$mm, $n = 15$, $m = 20$ (see Figure 3). This assumes that when $\theta = 0$ the pin starts at $r_1 + r_2$ along the x-axis.

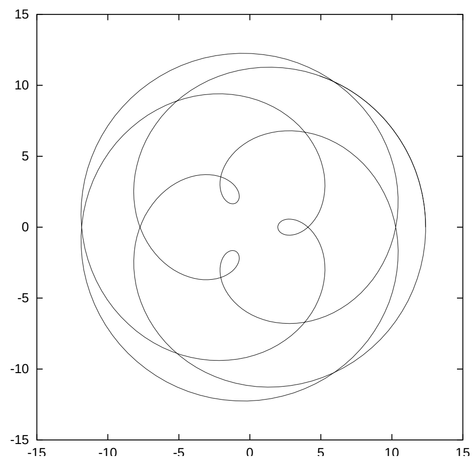

Figure 3: Epitrochoid

The path of any pin depends on its initial starting position. If it starts at $(\theta, \phi) = (\theta_i, \phi_i)$ then its path is given by

$$x = r_1 \cos(\theta + \theta_i) + r_2 \cos\left(\left(1 + \frac{n}{m}\right)\theta + \phi_i\right)$$
$$y = r_1 \sin(\theta + \theta_i) + r_2 \sin\left(\left(1 + \frac{n}{m}\right)\theta + \phi_i\right).$$

So the placement of a second pin on the same beater results in both pins tracing the same path only if the pins are diametrically opposite each other on the beater i.e. $(\theta_2, \phi_2) = (\theta_1, \phi_1 + \pi)$.

Pins on a second beater follow a different path in general, since $(\theta_3, \phi_3) = (\theta_1 + \pi, \phi_1 + \alpha)$ and $(\theta_4, \phi_4) = (\theta_1 + \pi, \phi_1 + \alpha + \pi)$. Now let $\theta_1 = \phi_1 = 0$ so that the first beater's pins are on the x-axis at $r_1 + r_2$ and $r_1 - r_2$. The two paths, traced out by each pair of pins on the two beaters, are identical only when $\alpha = \pm\pi/4, \pm 3\pi/4$ (see Figure 4). So the starting position of the four pins could, for example, be

$$(\theta_i, \phi_i) = (0, 0), (0, \pi), (\pi, \pi/4), (\pi, 5\pi/4),$$

in order to maximize the available "free" space. Note that this is probably the most important factor in positioning a pin in the bowl. If $\alpha = 0$ a beater pin (1.7mm diameter) would just clear each bowl pin (1.5mm diameter) - obviously an undesirable situation when there is dough of high viscosity between the two pins. From now on we assume that the beater pins are positioned optimally. There are five dissimilar regions in

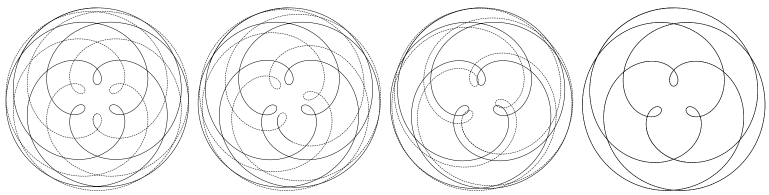

Figure 4: Centreline paths for 4 beater pins starting at $(\theta_i, \phi_i) = (0,0), (0,\pi), (\pi,\alpha), (\pi,\alpha+\pi)$ where $\alpha = 0, \pi/12, \pi/6, \pi/4$.

which one could place a 1.5mm pin in the bowl, see Figure 5. It turns out to be impossible to place a pin in Region 4 because of the thickness of the finite pin path. Regions 3 and 5 offer too little free space around the stationary pin, while region 1 is too close to the centre so that a pin here would not hold the dough as well as one placed in region 2. Let us therefore assume that the bowl pins are placed in the three sections of region 2.

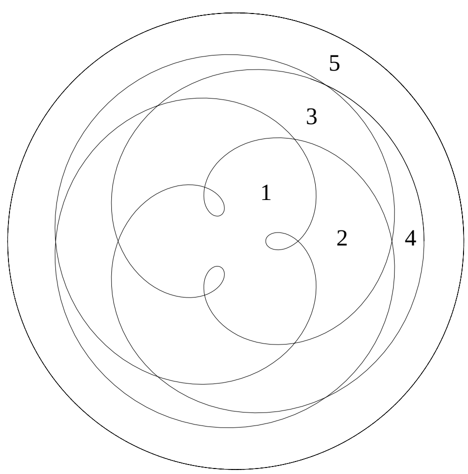

Figure 5: Possible bowl pin positions

x-axis crossing	θ	$x(\theta)$
1	0	12.382
2	2.432	-7.738
3	6.775	10.279
4	9.175	-11.847
5	11.771	4.096
6	12.566	1.968
7	13.362	4.097
8	15.958	-11.847
9	18.357	10.275
10	22.701	-7.739

Table 1: Zeros of $y(\theta)$

3 Approximate Solution

Since the epitrochoid is symmetric about the x-axis a bowl pin placed in region 2 would have its centre on the x-axis to maximise the nearest approach of a beater pin. The x position would (at least visually) be close to the midpoint of the two closest roots of the epitrochoid to region 2. To find the roots of the epitrochoid we need to find the roots of $y(\theta)$, shown in Figure 6. Note that starting from the position $(r, \theta) = (r_1 + r_2, 0)$ and

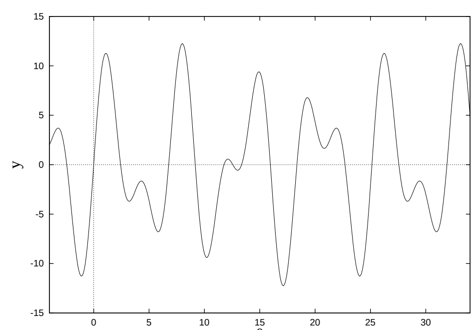

Figure 6: Graph of $y(\theta) = 7.1755 \sin\theta + 5.207 \sin(7\theta/4)$

following the epitrochoid in an anticlockwise direction a beater pin cuts the x-axis ten times before returning to its starting position. Finding the roots of $y(\theta)$ by Newton's Method and substituting into the equation for $x(\theta)$ leads to the results in the following table. The roots of interest are the third and fifth (also the seventh and ninth). So the x - position of the pin would be approximately $\frac{x(6.775)+x(11.771)}{2} = 7.18$ mm. Hence the bowl pins would be on a circle of radius approximately 7.18 mm at 120° spacing around the centre of the bowl.

4 Exact Solution

The exact geometric solution can be arrived at by considering the largest possible circle that can be placed in the free region so that it just touches the surrounding epitrochoid. Then a pin placed at the centre of this circle would be at the furthest point from any part of the epitrochoid.

Since the curve is symmetric about the x-axis we need only consider a circle with its centre on the x-axis. Consider two points moving anticlockwise around the epitrochoid starting from the seventh and ninth x-axis crossings respectively (see Figure 7). The positions of these two points

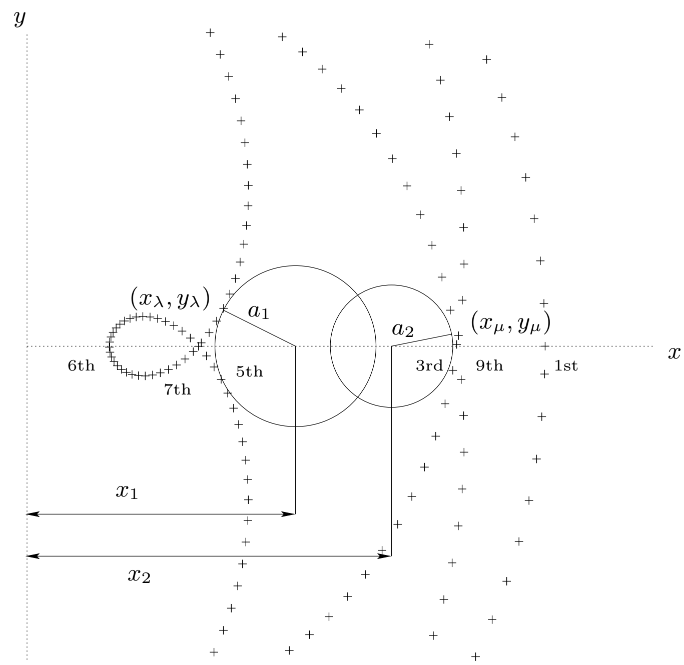

Figure 7: Maximizing circles

are completely determined by two angular parameters, λ and μ say, where

$$(x_\lambda, y_\lambda) = (r_1 \cos\lambda + r_2 \cos(7\lambda/4), r_1 \sin\lambda + r_2 \sin(7\lambda/4))$$
$$(x_\mu, y_\mu) = (r_1 \cos\mu + r_2 \cos(7\mu/4), r_1 \sin\mu + r_2 \sin(7\mu/4)).$$

The normal at each of these points would intersect the x-axis at the centre of a circle which would just touch the curve with two different radii a_1 and a_2. Requiring that these two circles be identical (i.e. $x_1 = x_2$ and $a_1 = a_2$) defines the position and size of the largest possible circle in this region. So now the normals are given by

$$[n_\lambda, m_\lambda] := \left[-\frac{dy_\lambda}{d\lambda}, \frac{dx_\lambda}{d\lambda}\right]; \qquad [n_\mu, m_\mu] := \left[-\frac{dy_\mu}{d\mu}, \frac{dx_\mu}{d\mu}\right].$$

So the lines through each parametric point in the direction of the respective normals are

$$(x, y) = (x_\lambda, y_\lambda) + s(n_\lambda, m_\lambda); \qquad (x, y) = (x_\mu, y_\mu) + t(n_\mu, m_\mu).$$

Intersecting these two lines with the x-axis gives

$$s = -\frac{y_\lambda}{m_\lambda} \quad \text{and} \quad t = -\frac{y_\mu}{m_\mu}$$

and so

$$x_1 = x_\lambda - \frac{n_\lambda}{m_\lambda} y_\lambda; \quad x_2 = x_\mu - \frac{n_\mu}{m_\mu} y_\mu.$$

Equating the resulting x positions gives the first relationship between λ and μ

$$x_\lambda - \frac{n_\lambda}{m_\lambda} y_\lambda = x_\mu - \frac{n_\mu}{m_\mu} y_\mu. \tag{2}$$

Now the radii of the two circles are given by

$$a_1^2 = (x_\lambda - x_1)^2 + (y_\lambda - 0)^2; \quad a_2^2 = (x_\mu - x_2)^2 + (y_\mu - 0)^2.$$

Equating these gives us our second equation relating λ and μ namely

$$\left(\frac{n_\lambda}{m_\lambda} y_\lambda\right)^2 + y_\lambda^2 = \left(\frac{n_\mu}{m_\mu} y_\mu\right)^2 + y_\mu^2. \tag{3}$$

Starting with λ and μ as the angles (in radians) corresponding to the seventh and ninth roots of $y(\theta)$ i.e $\lambda = 13.362$ and $\mu = 18.357$ one can use iteration to find the simultaneous solution to equations (2) and (3) as $\lambda = 13.60555$ and $\mu = 18.41436$. For these two values we get $a_1 = a_2 =$ 2.765mm and $r_{\text{pin}} = x_1 = x_2 = 7.435$mm. The free space around a 1.5mm diameter pin in the bowl with 1.7mm diameter pins on the beaters would be $a_1 - \frac{1.7}{2} - \frac{1.5}{2} = 1.165$mm. So even with all four beater pins travelling in the same path the clearance around a bowl pin is still less than one pin diameter.

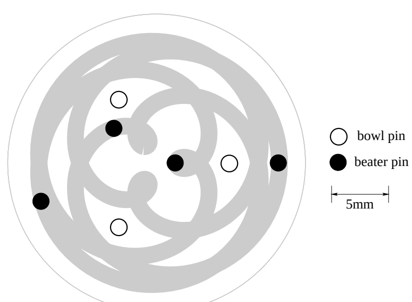

Figure 8: Starting position and path of all seven pins

5 Conclusion

The solution given above is, as already mentioned, the exact geometric solution of our problem. It has been used to create a prototype device which (through discussions with Chis Rath) has worked exceedingly well, considering the tolerances involved.

However, it does not take into account the forces exerted on the pins. Given a constant angular velocity of the drive gear, the beater pins have a higher linear velocity the further they are from the centre of the bowl. Consequently the forces on the pins (at closest approach) would be least when the beater pin is closer to the centre of the bowl than the bowl pin. Similarly, the forces would be greatest when the beater pin is further from centre than the bowl pin. This would suggest that the bowl pin should be shifted slightly closer to the centre of the bowl to balance these forces. However the forces will vary as the viscosity of the mixture changes and so this approach may not be a useful one in practice.

6 Acknowledgement

I would like to thank David Stafford for putting Chris Rath in touch with me and hence giving me the opportunity to solve such a delightful problem. Useful contributions were also made by Peter Thomas, Malcolm Park and Dr. Roger Eggleton.