Pseudopowerful Numbers

Ralph H. Buchholz

June 25, 2001

1 Introduction

The first time I encountered "powerful numbers" was in 1980 while reading an early Mathematical Games column by Martin Gardner (see [1]). Such numbers have the property that they are equal to the sum of the p-th power of each digit for some positive integer p, e.g. $3^3 + 7^3 + 1^3 = 371$.

A few years later I read an article (see [2]) on the Steinhaus Problem which described progress on the very same problem. Upon reading this article I conceived the following analogous problem.

"Do there exist positive integers equal to the sum of the digits of its p-th power?"

If we let $f_p(n) = \Sigma_{digits}(n^p)$ then we are looking for solutions to the equation $f_p(n) = n$. Clearly $f_p(0) = 0$ and $f_p(1) = 1$ for all $p > 0$ and hence, such numbers exist. I call them p-pseudopowerful (or just pseudopowerful) and immediately searched for non-trivial solutions turning up those in Table 1. Note

p	$n : \Sigma_{digits} n^p = n$
1	2,3,4,5,6,7,8,9
2	9
3	8,17,18,26,27
4	7,22,25,28,36
5	28,35,36,46
6	18,45,54,64
7	18,27,31,34,43,53,58,68
8	46,54,63
9	54,71,81
10	82,85,94,97,106,117

Table 1: Pseudopowerful numbers

that one could define[1] $\overline{f}_p(n) = (\Sigma_{digits} n)^p$ and find solutions to $\overline{f}_p(n) = n$. It turns out that there is a one to one correspondence between fixed points of f_p

[1] Which is precisely what I originally did. However the numbers became a tad large — hence the modification.

and those of $\overline{f}_p$. If $\overline{f}_p(n) = n$ then it is easy to show that $f_p(\Sigma_{digits} n) = \Sigma_{digits} n$. Similarly, if $f_p(n) = n$ then $\overline{f}_p(n^p) = n^p$, so it is sufficient to study f_p itself.

2 Main Theorems

The first useful fact one uncovers is that, just as for powerful numbers, there are only finitely many pseudopowerful numbers for each given exponent p.

Theorem 1 *There exist no pseudopowerful numbers for* $n > n_{\max} = 9r_\infty$ *where* r_∞ *is defined by* $r_\infty = 1 + p \log_{10}(9r_\infty)$.

Proof: Suppose we let the r digits of n^p be denoted by $d_0, d_1, \ldots, d_{r-1}$ where d_0 is the least significant digit. Then the defining equation for a p-pseudopowerful number, namely $f_p(n) = n$, is equivalent to

$$(d_0 + d_1 + \ldots + d_{r-1})^p = d_0 + 10d_1 + \ldots + 10^{r-1} d_{r-1}. \quad (1)$$

If we assume that d_{r-1} is non-zero, so that we have a proper r-digit number, then the maximum that the left-hand-side of equation (1) can reach occurs when all the digits are nine, i.e. $(9r)^p$. Meanwhile the minimum of the right-hand-side of equation (1) is 10^{r-1}. Clearly, for a fixed p we have

$$(9r)^p < 10^{r-1}$$

for sufficiently large r so that equation (1) is never solvable if $r > r_\infty$ where r_∞ is the solution to $(9r_\infty)^p = 10^{r_\infty - 1}$. **QED**

The form of the defining equation for r_∞ lends itself well to an iterative solution technique. Thus for each exponent p one need simply evaluate r_∞ and then check each integer from 2 up to $9r_\infty$ to find all p-pseudopowerful numbers. It is possible to improve this finite search by using the following result.
Theorem 2 *If* n *is* p*-pseudopowerful then* $n^p \equiv n \pmod 9$.

Proof: The result follows by simply considering equation (1) modulo 9 and observing that $n = d_0 + d_1 + \ldots + d_{r-1}$. **QED**

The point is that for each chosen p we need only consider the restricted values of n which satisfy Theorem 2. For example, if $p = 2$, then we need only check values of n which are zero or one modulo 9. Similarly, since $\phi(9) = 6$ the modulo 6 character of p will tell us precisely which values of n can satisfy $n^p \equiv n \pmod 9$ as in Table 2 and hence possibly be pseudopowerful. Using Theorem 2 one can eliminate about 61% of the integers in the range $n \in [2, \ldots, 9r_\infty]$ with the resultant cutdown in search time. The results given in Tables 4, 5, and 6 of the Appendix show the erratic behaviour of the function

$$\nu(p) := \#\{n : f_p(n) = n\} - 2$$

namely the number of non-trivial pseudopowerfuls for each p.

$p \pmod 6) : p \geq 2$	$n : n^p \equiv n \pmod 9$
0	0,1
1	0,1,2,4,5,7,8
2	0,1
3	0,1,8
4	0,1,4,7
5	0,1,8

Table 2: Modular restrictions

3 Open Questions

A number of interesting questions arose early in the exploration of this problem.

Do there exist exponents for which there are no non-trivial pseudopowerful numbers? Yes, since $\nu(105) = 0$. Do there exist infinitely many such exponents? Probably not! Since the number of possibilities increases with each p.

Next, if we consider the number of modular solutions in Table 2, we ask if it is possible that $\min\{\nu(p) : p \equiv 1 \pmod 6\} \geq \max\{\nu(p) : p \not\equiv 1 \pmod 6\}$. Despite the fact that it holds for all of Table 4 it does not hold in general, since $\nu(55) = 2$ while $\nu(54) = 5$.

Is the number of solutions bounded independently of p? This is not as implausible as it seems at first sight. If we model f_p as a random mapping then the expected number of fixed points is one. Furthermore, the maximum number of solutions, namely 13, has already occurred as early as $p = 25$.

Can a pseudopowerful number equal the sum of two distinct pseudopowerful numbers? This leads to the equation

$$\Sigma_{digits}(a^p) + \Sigma_{digits}(b^p) = \Sigma_{digits}(c^p)$$

where $f_p(a) = a$, $f_p(b) = b$, $f_p(c) = c$ which is reminiscent of Fermat's equation. It is possible to find non-trivial solutions to this equation for a number of values of p which are shown in Table 3. Attempting to determine whether or not

p	$(a, b, c) : f_p(a) + f_p(b) = f_p(c)$
1	(2,3,5),(2,4,6),(2,5,7),(2,6,8),(2,7,9)
	(3,4,7),(3,5,8),(3,6,9),(4,5,9)
3	(8,18,26)
7	(27,31,58)
13	(20,86,106),(20,106,126),(20,126,146),(40,86,126),(40,106,146)

Table 3: Pseudopowerful numbers satisfying $a + b = c$

these are the only solutions would require good lower and upper bounds on p-pseudopowerful numbers — which I do not yet have in hand. Note that if we replace $f_p(x)$ by $\overline{f}_p(x)$ then there are no solutions since we are led to the equation

$$(\Sigma_{digits} a)^p + (\Sigma_{digits} b)^p = (\Sigma_{digits} c)^p$$

which is impossible by Wiles' proof of Fermat's Last theorem.

4 References

[1] *Mathematical Games*, Martin Gardner, Scientific American, January, 1963.
[2] *Numbers Count*, M.R. Mudge, Australian Personal Computing, p.102, April 1983.

p	n_{max}	$n : \Sigma_{digits} n^p = n$
1	30	2,3,4,5,6,7,8,9
2	57	9
3	86	8,17,18,26,27
4	117	7,22,25,28,36
5	149	28,35,36,46
6	182	18,45,54,64
7	216	18,27,31,34,43,53,58,68
8	250	46,54,63
9	285	54,71,81
10	320	82,85,94,97,106,117
11	355	98,107,108
12	392	108
13	428	20,40,86,103,104,106,107,126,134,135,146
14	465	91,118,127,135,154
15	502	107,134,136,152,154,172,199
16	539	133,142,163,169,181,187
17	577	80,143,171,216
18	615	172,181
19	653	80,90,155,157,171,173,181,189,207
20	691	90,181,207
21	730	90,199,225
22	769	90,169,193,217,225,234,256
23	808	234,244,271
24	847	252,262,288
25	886	140,211,221,236,256,257,261,277,295,296,298,299,337
26	926	306,307,316,324
27	966	305,307
28	1006	90,160,265,292,301,328
29	1046	305,314,325,332,341
30	1086	396
31	1126	170,331,338,346,356,364,367,386,387,443
32	1167	388
33	1207	170,352,359,378,406,422,423
34	1248	387,412,463
35	1289	378,388,414,451,477
36	1330	388,424
37	1371	414,421,422,433,469,477,485,495
38	1412	468,469
39	1453	449,523
40	1495	250,441,468,486,495,502

Table 4: Pseudopowerful numbers for $p = 1 \ldots 40$

p	n_{max}	$n : \Sigma_{digits} n^p = n$
41	1536	432
42	1578	280,487,523,531
43	1620	461,499,508,511,526,532,542,548,572
44	1662	280,523,549,576,603
45	1704	360,503,523
46	1746	360,478,514,522,544,558,574,592
47	1788	350,559,567,575,576,595,603,666
48	1830	370,513,631,667
49	1872	270,290,340,350,360,533,589,637,648,661,695
50	1915	685
51	1957	360,666,685
52	2000	625,688,736,739
53	2043	648,683,703,746
54	2085	370,603,657,667,739
55	2128	677,683
56	2171	684
57	2214	370,460,719,748,793,802
58	2257	667,721,754
59	2300	370,440,693,845
60	2343	694,784,792,793
61	2387	440,490,758,815,833
62	2430	855,865
63	2474	827,836,846
64	2517	430,793,829,871
65	2561	818,856,891,928
66	2604	837,864,927
67	2648	450,859,865,866,869,874,926,934
68	2692	837
69	2735	540,936,962,963,1016
70	2779	540,882,909
71	2823	917,991
72	2867	901,1062
73	2911	853,882,928,1006,1015
74	2955	936,1008,1009,1018
75	3000	630,964,999,1016,1053
76	3044	1044,1075,1093
77	3088	1061,1062,1088
78	3132	964,1117,1126,1134
79	3177	610,1031,1043,1054,1064,1091,1108,1133
80	3221	1044,1071,1134,1144

Table 5: Pseudopowerful numbers for $p = 41 \ldots 80$

p	n_{max}	$n : \Sigma_{digits} n^p = n$
81	3266	1062,1196
82	3310	1048,1111,1134,1231
83	3355	730,1115,1151,1207
84	3400	1188
85	3444	1051,1103,1165,1183,1277
86	3489	1134,1225
87	3534	1187,1216,1224,1232,1278,1288
88	3579	730,1084,1147,1183,1186,1206
89	3624	1151,1232,1358
90	3669	1306,1422
91	3714	720,1208,1233,1253,1258,1261,1278
92	3759	720,1296,1359
93	3804	810,820,1396
94	3849	1285,1287,1303,1327,1332,1339,1341,1444
95	3894	820,1323,1342,1351,1385
96	3939	1387
97	3985	1237,1322,1324,1361,1367,1397,1442
98	4030	1359
99	4075	1322,1403,1405,1441
100	4121	1363,1378,1408,1414,1489
101	4166	1423,1468
102	4212	1359,1432,1611
103	4257	1379,1445,1476,1477,1484,1486,1495,1496,1523
104	4303	1377,1476
105	4348	—
106	4394	1444,1456,1458,1474,1546,1552,1558,1567,1573
107	4440	1574,1691
108	4486	1486,1621,1639,1648
109	4531	1507,1523,1562,1565,1585,1603,1628,1642
110	4577	1459
111	4623	910,1539,1548,1647,1682
112	4669	990,1030,1504,1519
113	4715	1548,1674,1738
114	4761	1521
115	4807	1080,1526,1546,1553,1634,1636,1656,1684,1714,1717,1823
116	4853	1621,1647,1693
117	4899	1773
118	4945	1674,1764
119	4991	1665,1673
120	5037	1657,1702

Table 6: Pseudopowerful numbers for $p = 81 \ldots 120$