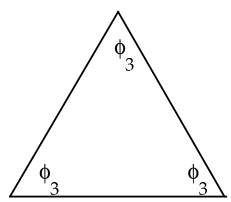
Polygons with Integral Internal Angles

Ralph H. Buchholz

July 1986 - SMJ 33

Consider the simplest polygon the equilateral triangle with unit edges and equal internal angles ϕ_3 say. Since the sum of the angles of any triangle must

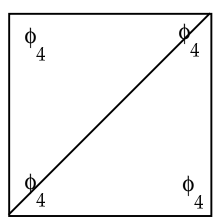


be 180° we see that

$$3\phi_3 = 180^\circ$$

or $\phi_3 = 60^\circ$ which is clearly a whole number of degrees.

Next consider a square which has four equal internal angles. Note that it can be decomposed into two triangles as in Figure 2. Since the sum of all the



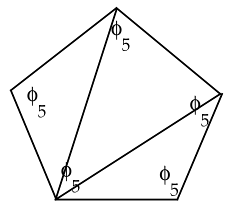
internal angles of the square must equal the sum of the internal angles of each separate triangle we see that

$$4\phi_4 = 2 \times 180^\circ$$

or $\phi_4 = 90^\circ$.

Similarly a pentagon can be decomposed into three triangles so that

$$5\phi_5 = 3 \times 180^\circ$$



or $\phi_5 = 108°$, again an integer. By now one should be able to see the emerging pattern that in general a regular n-gon can be decomposed into $(n-2)$ non-overlapping triangles each with an angle sum of 180°. Since a regular n-gon also has n identical internal angles, denoted by ϕ_n say, we find that

$$\begin{aligned} n\phi_n &= (n-2)180° \text{or} \\ \phi_n &= \frac{(n-2)}{n} \cdot 180° \end{aligned} \tag{1}$$

Now note that ϕ_n is not always an integer since for a regular heptagon we get

$$\begin{aligned} \phi_7 &= \frac{(7-2)}{7} \cdot 180° \text{or} \\ \phi_7 &= 128\frac{4}{7}^{\circ} \end{aligned}$$

Problem : Are there an infinite number of polygons with integral ϕ_n? Answer : Since the internal angle of any regular polygon must be less than and not equal to 180° we see immediately that there can only be a finite number of such polygons as the largest possible integral internal angle is 179°.

In fact from equation 1 above we see that ϕ_n can be an integer if and only if n divides $(n-2)180°$. For a convenient shorthand the statement "a divides b" is usually denoted by "a—b". From now on we shall drop the degree symbol.

$$\begin{aligned} \text{So} \quad & n|(n-2)180 \\ \text{iff} \quad & n|(180n-360) \\ \text{iff} \quad & n|-360 \\ \text{iff} \quad & n|360 \\ \text{iff} \quad & n|2^3 3^2 5. \end{aligned}$$

From this last statement you should be able to systematically verify that the only positive dividers of 360 must be a member of the following set :-

$$\{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360\}.$$

Since the equilateral triangle is the smallest polygon, n must be greater than or equal to 3 and so only the latter 22 elements of the above set lead to a polygon with integral internal angles as shown in the table below.

n	ϕ_n	n	ϕ_n
3	60	24	165
4	90	30	168
5	108	36	176
6	120	40	171
8	135	45	172
9	140	60	174
10	144	72	175
12	150	90	176
15	156	120	177
18	160	180	178
20	162	360	179

Table 1: Regular n-gons with integral ϕ_n