Spiral Polygon Series

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Consider the two objects in Figure 1. On the left we see three equal squares made up of 12 unit edges while on the right we have four equal squares made up of 12 unit edges some of which are “shared”.

Figure 1: Squares with 12 edges

This is in fact the basis of a well known problem in which one is asked to convert one configuration into the other by moving only three edges.

Generalising this concept let us examine the following two problems :-

(a) What is the maximum number $S_{max}(n)$ of unit edges required to construct n equal squares with no edges left over, and

(b) What is the minimum number $S_{min}(n)$ of unit edges required to construct n equal squares if sharing (of edges) is allowed.

Clearly to maximise the number of edges we should just construct n disjoint squares each with 4 edges and thus $S_{max}(n) = 4n$.

However to minimise the number of edges is not quite as easy. Consider the series of objects in Figure 2 which show a minimum configuration of edges to construct one to twelve squares. For the first square we can do no better than 4 unit edges. For the second and third squares we can share at most 1 edge each thus requiring 3 extra edges each. But for the fourth square we can share 2 edges thus requiring only two extra edges. Continuing in this manner we see that to obtain a minimising configuration for each successive square simply add a square in a spiral pattern. Note that some minimum configurations are not unique, for example three squares can also be constructed with 10 edges as in Figure 3. However, in general for n squares we must seek to minimise the perimeter (as the internal area will remain the same for different configurations of n squares). This can best be achieved by combining the squares to form a pattern as close as possible to a circle, hence the spiral algorithm (shown in Figure 4). Now to obtain the explicit formula for $S_{min}(n)$ consider the case

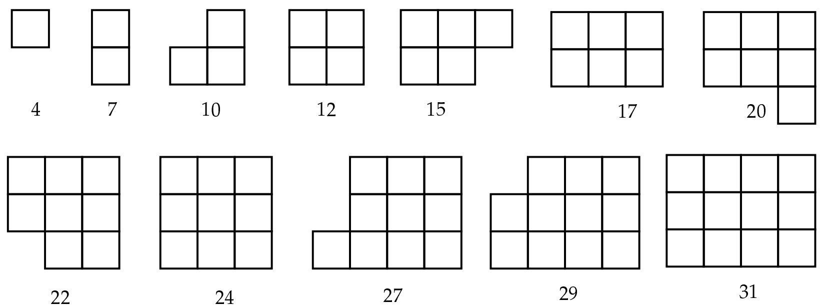

Figure 2: Minimum Edges for 1 to 12 squares

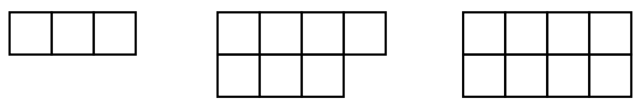

Figure 3: Alternate minimum configurations for 3,7 and 8 squares

$n = m^2$. Here the minimum configuration is always a complete large square made up of $m \times m$ unit squares.

Referring to Figure 5 we see that the number of vertical unit edges is $(m + 1) \times m$ and similarly for the number of horizontal edges.

$$\text{Thus } S_{min}(m^2) = (m+1)m + (m+1)m$$
$$= 2m^2 + 2m.$$

Now with this as a basis and assuming that the spiral algorithm does in fact provide the minimum configuration for every positive n one can prove that :-

$$S_{min}(n) = \lfloor 2n + 2\sqrt{n} \rfloor$$

where $\lfloor x \rfloor$denotes the least integer greater than or equal to x. For example

$$S_{min}(12) = \lfloor 24 + 2\sqrt{12} \rfloor$$
$$= \lfloor 30.928 \rfloor$$
$$= 31. \text{ (c.f. Figure 2)}$$

Since both the equilateral triangle and regular hexagon can tile the infinite plane we can pose analagous questions to those for the square. Again we find that for both polygons a spiral pattern provides a minimum solution at each step (see Figures 6 and 7).

As before we can extract the explicit formula for these two algorithms by first considering the results for each completed hexagon. For the equilateral

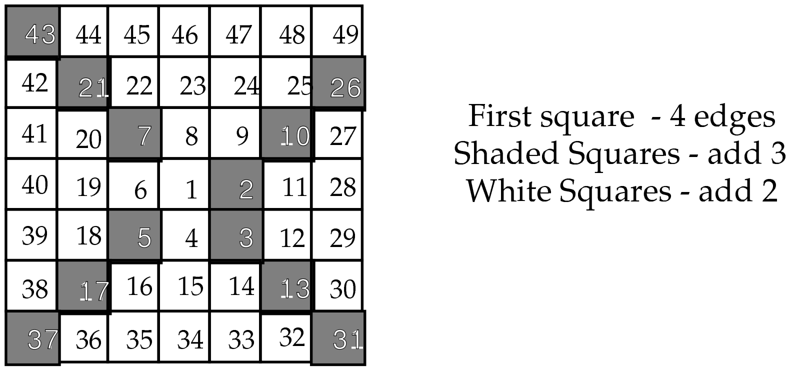

Figure 4: Square Spiral Algorithm

triangle we have

$$E_{max}(n) = 3n$$
$$E_{min}(n) = \lfloor \frac{3n}{2} + \sqrt{\frac{3n}{4}} \rfloor.$$

For the regular hexagon we have

$$H_{max}(n) = 6n$$
$$H_{min}(n) = \lfloor 3n + \sqrt{12n - 3} \rfloor.$$

Are these the only regular polygons for which we can pose this problem? What if we relax the "regular" restriction.

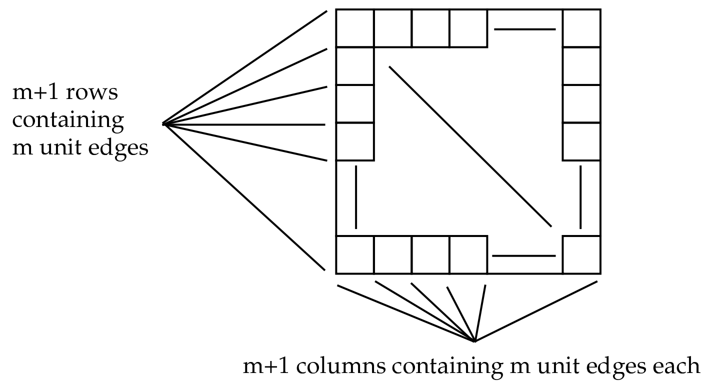

Figure 5: Square Spiral Algorithm for perfect squares

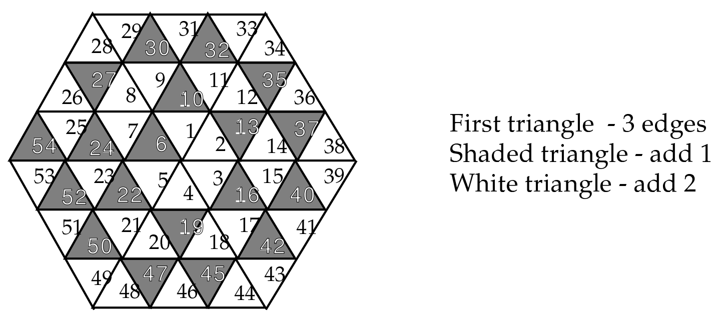

Figure 6: Triangular Spiral Algorithm

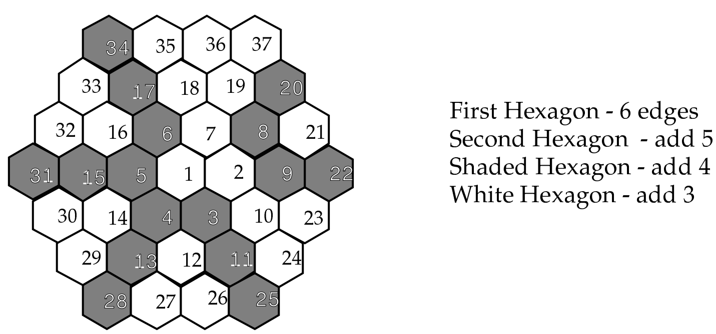

Figure 7: Hexagonal Spiral Algorithm