Square-like triangles

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Abstract

We demonstrate that there are infinitely many triangles with the property that their area is the square of the tierce-semi-perimeter. These turn out to correspond to rational points on a rank one elliptic curve.

Keywords : Heron triangle, elliptic curve.

1 Introduction

If one considers the set of rational-sided rectangles, then a simple calculation reveals that the square is the only type which has the property that its area is equal to the square of half its semiperimeter. Clearly, if a rectangle has sides a and b then its semiperimeter is $s = a + b$ and

$$ab = \left(\frac{s}{2}\right)^2 \quad \text{implies} \quad a = b.$$

Now since the perimeter-to-area ratio is larger for triangles than it is for squares it seems appropriate to adjust the condition to ask if we can find solutions to

$$\text{Area}(a, b, c) = \left(\frac{s}{3}\right)^2. \tag{1}$$

An example of such a triangle is $(a, b, c) = (9, 10, 17)$ with area 36 and semi-perimeter 18 so that $36 = (18/3)^2$.

2 Transformation to an elliptic curve

If we let $\Delta(a, b, c)$ denote the area of a triangle with sides a, b, c then we immediately recall what Hero of Alexandria taught us about 2000 years ago, namely,

$$\Delta(a, b, c) = \sqrt{s(s-a)(s-b)(s-c)}$$

where s is the semiperimeter of the triangle. Using this we can rewrite equation (1) as

$$s(s-a)(s-b)(s-c) = \left(\frac{s}{3}\right)^4$$

or equivalently, since $s \neq 0$,

$$3^4(s-a)(s-b)(s-c) = s^3.$$

We find it useful to consider this as a homogeneous curve, $F = 0$, where F is defined by

$$F = 81(-a+b+c)(a-b+c)(a+b-c) - (a+b+c)^3. \qquad (2)$$

It was instructive (at least for the author) to follow the path provided by Knapp [2] to convert this into an elliptic curve in Weierstrass normal form. The first step is to find a rational flex. All flexes are determined by finding the solutions of the equation

$$\text{Resultant}(F, |H(F)|) = 0,$$

where $H(F)$ is the Hessian matrix of second partial derivatives given by

$$H(F) = \begin{bmatrix} F_{aa} & F_{ab} & F_{ac} \\ F_{ba} & F_{bb} & F_{bc} \\ F_{ca} & F_{cb} & F_{cc} \end{bmatrix}.$$

Thus we calculate

$$\frac{|H(F)|}{2^6 3^9} = -11(a^3+b^3+c^3) + 13(a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2)) - 36abc$$

and hence obtain

$$\text{Resultant}(F, |H(F)|) = 2^{32}3^{31}bc(b+c)p_6(b,c)$$

where

$$p_6 = 169c^6 - 897c^5b + 2094c^4b^2 - 2705c^3b^3 + 2094c^2b^4 - 897cb^5 + 169b^6.$$

We could pause here to consider the solutions to $p_6 = 0$ but it is a trivial exercise to show that there are no non-trivial rational ones. So we just ignore it and conclude that F and $|H(F)|$ have a common zero, and hence a point of inflexion when $b + c = 0$, namely at $(a,b,c) = (0,1,-1)$. The next step is to move this point to the point at infinity in $\mathbb{P}^2(\mathbb{C})$, namely $(0,1,0)$. This is accomplished through the judicious choice of a matrix ϕ_1 with determinant 1 such that $\phi_1(0,1,-1) = (0,1,0)$. Clearly,

$$\phi_1 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

does the job nicely. The transformed curve, which Knapp calls F^{ϕ_1}, is given by

$$\begin{aligned} F^{\phi_1} &= F(\phi_1^{-1}(a,b,c)) \\ &= F(c,b,a-b) \\ &= -82(a^3+c^3)+78ac(a+c)+324b(b-a)(c-a). \end{aligned}$$

The tangent line at the point at infinity is given by

$$\left[\frac{\partial F^{\phi_1}}{\partial a}\right]_{[0,1,0]} a + \left[\frac{\partial F^{\phi_1}}{\partial b}\right]_{[0,1,0]} b + \left[\frac{\partial F^{\phi_1}}{\partial c}\right]_{[0,1,0]} c = 0$$

or $4a - 4c = 0$. We can twist this line to $c = 0$ by setting

$$\phi_2^{-1} = \begin{bmatrix} a & 0 & p \\ 0 & 1 & 0 \\ c & 0 & q \end{bmatrix}$$

where a and c are solutions to the tangent equation and p and q are chosen so that ϕ_2^{-1} is non-singular. We choose

$$\phi_2^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

so that

$$\begin{aligned} (F^{\phi_1})^{\phi_2} &= F^{\phi_1}(\phi_2^{-1}(a,b,c)) \\ &= F^{\phi_1}(a+c,b,a-c) \\ &= 8(-81b^2c+81abc+81bc^2-81ac^2-a^3). \end{aligned}$$

Finally, we simply need to ensure that the coefficients of b^2c and a^3 in $(F^{\phi_1})^{\phi_2}$ are the same magnitude and opposite sign. If we set

$$\phi_3^{-1} = \begin{bmatrix} -t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $t = C_{bbc}/C_{aaa} = -648/-8 = 81$ then we get

$$\begin{aligned} ((F^{\phi_1})^{\phi_2})^{\phi_3} &= (F^{\phi_1})^{\phi_2}(\phi_3^{-1}(a,b,c)) \\ &= (F^{\phi_1})^{\phi_2}(-81a,81b,c) \\ &= 2^3 3^8(-81abc+ac^2-81b^2c+81a^3+bc^2). \end{aligned}$$

Thus we have, if we dehomogenise at c, the elliptic curve

$$E : y^2 + (x-1/81)y = x^3 + x/81$$

and the complete transformation, $\Phi : E \mapsto F$, is given by

$$\begin{aligned}(a,b,c) &:= \Phi(x,y)\\ &= \phi_1^{-1}(\phi_2^{-1}(\phi_3^{-1}(x,y,1)))\\ &= (-81x-1, 81y, -81x-81y+1).\end{aligned}$$

Now using the `Magma` package one easily finds that the group of rational points $E(\mathbb{Q})$ is isomorphic to the group $\frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \mathbb{Z}$ and in fact

$$E(\mathbb{Q}) = \langle T, P \rangle$$

where $T = (0, 1/81)$ and $P = (-1/36, 17/648)$. Using the mapping Φ we find that $\Phi(P) = [10/8, 17/8, 9/8]$ which when scaled becomes the example mentioned in the introduction. Similarly, we can enumerate as many such triangles as we wish by simply considering odd multiples of the infinite order point, P, as shown in Table 1. Note the rather large sizes of the sides of the second and

n	sides
1	[10,17,9]
3	[219306474,103852745,142121521]
5	[1541659583556321311 0186, 2579483403038729537 8353, 3647774730837390379 0345]
7	[597303897018993364059085208046971311117 84778, 781540243534911804739107332470015594476 83401, 295755675895709741062989984320910157253 06385]

Table 1: Square-like triangles corresponding to odd multiples of P

subsequent trinagles.

3 Generalisations

If we vary the fraction of the semiperimeter so that we ask for solutions to the equation

$$\Delta(a,b,c) = \left(\frac{s}{n}\right)^2 \tag{3}$$

then, by following the analogous transformation path as that in the previous section we find that, this corresponds to the family of elliptic curves

$$E_n : y^2 + xy - \left(\frac{1}{n}\right)^4 y = x^3 + \left(\frac{1}{n}\right)^4 x.$$

Again using `Magma` to determine the group $E_n(\mathbb{Q})$ leads to the ranks shown in Table 2. An alternate generalisation is to consider cyclic quadrilaterals with the

n	rank	torsion
1	0	$\mathbb{Z}/3\mathbb{Z}$
2	0	$\mathbb{Z}/3\mathbb{Z}$
3	1	$\mathbb{Z}/3\mathbb{Z}$
4	1	$\mathbb{Z}/3\mathbb{Z}$
5	1	$\mathbb{Z}/3\mathbb{Z}$
6	0	$\mathbb{Z}/3\mathbb{Z}$
7	1	$\mathbb{Z}/3\mathbb{Z}$

Table 2: Rank of elliptic curves $E_n(\mathbb{Q})$

property that

$$\texttt{Area}(a,b,c,d) = \left(\frac{s}{2}\right)^2$$

which leads to consideration of the equation

$$16(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d) = (a+b+c+d)^4. \quad (4)$$

This quartic surface in four variables is not amenable to the techniques used for the triangular problem and so at present the best we can do is to attempt to find a solution through computational means. Unfortunately, the only rational d satisfying equation 4 that have turned up, with a, b, c inside the box $a \le b \le c \le 500$, were those in which $a = b = c = d$.

One could, of course, consider other cyclic n-gons (see [1]) or even higher dimensional variants—but we leave these for the future.

4 References

1. Ralph H. Buchholz and James A. MacDougall, *Cyclic polygons with rational sides and area*, submitted.
2. Knapp, *Elliptic Curves*
3. `http://www.hyperdictionary.com/`