The square, the triangle and the hexagon

Ralph H. Buchholz and Warwick de Launey

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Abstract

In this paper we will examine the following problem: What is the minimum number of unit edges required to construct k identical size regular polygons in the plane if sharing of edges is allowed?

1 Introduction

In this paper we will examine the following problem:

Question 1 *What is the minimum number of sides required to construct k identical size regular polygons in the plane if sharing of sides is allowed?*

In the world of mathematics, there are just three regular polygons which tessellate the plane: the square, equilateral triangle, and regular hexagon. We will answer Question 1 for these shapes. This had already been done by Harary and Harborth [3] however, we had not found that reference until later[1]. As usual when one finds one has been beaten to the punch we hope that this may be of some value.

1.1 The Square

Consider the series of objects in Figure 1 which show a minimal configuration of edges to construct one to twelve squares. For the first square we can do no better than 4 unit edges. For the second and third squares we can share at most 1 edge each thus requiring three extra edges each. But for the fourth square we can share two edges thus requiring only two extra edges. Note that some minimal configurations are not unique, for example three squares can also be constructed with 10 edges as in Figure 2. Notice that each edge appears in either one or two squares. Let $S(n,p)$ denote the number of edges in a configuration with n squares and p edges on the perimeter. Then

$$S(n,p) = \frac{4n-p}{2} + p$$

[1] In fact, not until 2008.

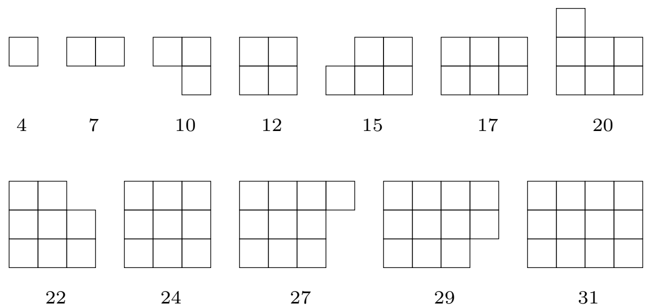


Figure 1: Minimal edge configurations for 1 to 12 squares

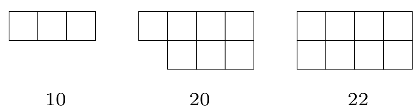


Figure 2: Alternate minimal configurations for 3,7 and 8 squares

$$= 2n + p/2.$$

Clearly, for a fixed number of squares the only way to minimise the number of edges is to minimise the perimeter. If we let $S_{\text{min}}(n)$ denote the minimum number of edges required to construct n unit squares then,

$$S_{\text{min}}(n) = \min_{p \leq 2n+2} \{S(n,p)\}.$$

Now a lower bound on the perimeter is given by a circle of area n. Thus we have $n = \pi r^2$ and $p \geq 2\pi r$ implies that $p \geq 2\sqrt{\pi n}$. So

$$S_{\text{min}}(n) \geq 2n + \sqrt{\pi n}.$$

Furthermore, since a square is the shape which minimises the perimeter of fixed area rectangles one sees that for $n = m^2$ we have

$$S_{\text{min}}(m^2) = 2m^2 + 2m.$$

At this stage one conjectures that the addition of the next square in a spiral pattern produces a minimal configuration for each n. To prove that this is in fact the case first note that for any particular (not necessarily convex) collection of squares, the perimeter is greater than or equal to that of the smallest aligned rectangle which can be drawn around it. Thus in Figure 3 we see that the perimeter of the solid configuration is $\geq 2a + 2b$. This is clear since the bounding dashed rectangle is oriented identically to the square in the solid configuration. With the reduction of an arbitrary configuration to its bounding

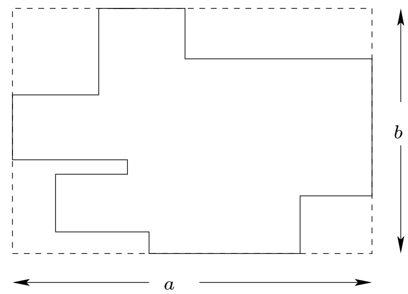


Figure 3: Minimal containing rectangle

rectangle now clear, all that is required is to make a comparison between a rectangular configuration and a corresponding configuration, with the same number of squares, obtained from the spiral algorithm.

Lemma 1 *Any rectangle, of size ab, containing the same number of unit squares as some configuration obtained using the spiral algorithm, has a perimeter greater than or equal to that of the perimeter of the spiral configuration.*

Proof: In the spiral algorithm an L-shaped gnomon of unit squares takes us from one completed square to the next. We break the proof in two subcases which correspond to the first and second legs of the gnomon respectively.

Case (i). If $ab = m^2 + t$, where $1 \leq t \leq m$ then

$$\begin{aligned}
&(\sqrt{m^2+t} - a)(\sqrt{m^2+t} - b) < 0 \\
&m^2 + t - (a+b)\sqrt{m^2+t} + ab < 0 \\
&(a+b)\sqrt{m^2+t} > 2(m^2+t) \\
&a + b > 2\sqrt{m^2+t} \\
&a + b > 2m.
\end{aligned}$$

Since $2(a+b) \geq 4m+2$, the rectangles perimeter is no smaller than the perimeter of the corresponding spiral configuration.

Case (ii). If $ab = m(m+1) + t$, where $1 \leq t \leq m+1$ then

$$\begin{aligned}
&(\sqrt{m(m+1)+t} - a)(\sqrt{m(m+1)+t} - b) < 0 \\
&m(m+1) + t - (a+b)\sqrt{m(m+1)+t} + ab < 0 \\
&(a+b)\sqrt{m(m+1)+t} > 2(m(m+1)+t) \\
&a + b > 2\sqrt{m(m+1)+1} \\
&a + b > 2m + 1.
\end{aligned}$$

So $2(a+b) \geq 4m+3$ and again the rectangles perimeter is no smaller than that of the spiral configuration. □

Hence the spiral algorithm (Figure 4) always does at least as well as any other rectangular configuration and so provides a minimal configuration for any number of squares. In fact,

Proposition 1 $S_{min}(n) = \lceil 2n + 2\sqrt{n} \rceil$.

43	44	45	46	47	48	49
42	21	22	23	24	25	26
41	20	7	8	9	10	27
40	19	6	1	2	11	28
39	18	5	4	3	12	29
38	17	16	15	14	13	30
37	36	35	34	33	32	31

First square - 4 edges

Shaded square - add 3 edges

White square - add 2 edges

Figure 4: Square spiral algorithm

Finally, for squares it is not too hard to characterize all minimal configurations. How thin can a rectangle be and still be a minimal configuration? First recall that a square with m unit squares on a side which has t unit squares added in the next layer has a perimeter given by

$$p_{square+t} = \begin{cases} 4m & \text{for } t = 0 \\ 4m+2 & \text{for } 1 \le t \le m \\ 4m+4 & \text{for } m+1 \le t \le 2m+1. \end{cases}$$

So we are looking for solutions to the simultaneous equations

$$n_{rectangle} = n_{square+t}$$

$$p_{rectangle} = p_{square+t}.$$

Case 1. If $t = 0$ then we have $a + b = 2m$ and $ab = m^2$ which have the unique solution $a = b = m$ and so there are no other minimal configurations.

Case 2. If $1 \le t \le m$ then we have the pair of equations

$$2(a+b) = 4m + 2$$

$$ab = m^2 + t.$$

Eliminating b by substituting the second equation into the first leads to the quadratic equation

$$a^2 - (2m+1)a + (m^2 + t)$$

which has the solutions

$$\begin{aligned} a &= \frac{2m + 1 \pm \sqrt{(2m+1)^2 - 4(m^2+t)}}{2} \\ &= \frac{2m+1 \pm \sqrt{4m - 4t + 1}}{2}. \end{aligned}$$

Thus any factorisation of $m^2 + t$ into ab such that

$$\frac{2m + 1 - \sqrt{4m - 4t + 1}}{2} \leq a \leq \frac{2m + 1 + \sqrt{4m - 4t + 1}}{2}$$

leads to a minimal rectangle.

Case 3. If $m + 1 \leq t \leq 2m + 1$ then we have the pair of equations

$$2(a + b) = 4m + 4$$

$$ab = m^2 + t.$$

These have the solutions

$$a = m + 1 \pm \sqrt{2m - t + 1}.$$

Thus any factorisation of $m^2 + t$ into ab such that

$$m + 1 - \sqrt{2m - t + 1} \leq a \leq m + 1 + \sqrt{2m - t + 1}$$

leads to a minimal rectangle in this case.

For example for a collection of 5016 unit squares we have $5016 = 70^2 + 116$ and since $116 > 70$ we are in case 2. Thus $m = 70$, $t = 116$ and we calculate

$$71 - \sqrt{25} \leq a \leq 71 + \sqrt{25}$$

$$66 \leq a \leq 76.$$

Hence a rectangle 66 by 76 has the same perimeter as a 70 by 70 square with 116 unit squares in the next layer. However a 57 by 88, 44 by 114 or thinner rectangles are not minimal.

1.2 Equilateral Triangles

Since both the equilateral triangle and regular hexagon can tile the infinite plane we can pose analogous questions to those for the square.

Let $E(n, p)$ denote the number of edges contained in a configuration of n equilateral triangles with p edges along the perimeter. Then

$$\begin{aligned} E(n, p) &= \frac{3n - p}{2} + p \\ &= \frac{3}{2}n + \frac{p}{2}. \end{aligned}$$

So, as before, one needs to minimise the perimeter to minimise the number of edges. Comparison with a circle gives us the lower bound

$$E_{\min}(n) \geq \frac{3}{2}n + \sqrt{\frac{\sqrt{3}\pi n}{4}}.$$

As before, we note that any configuration of triangles can be surrounded by a minimal aligned irregular hexagon, I say, such that the perimeter of the configuration is greater than or equal to that of the hexagon (see Figure 5). Note that we have the relationships

$$\begin{aligned} a+b &= d+e \\ a+f &= c+d \\ e+f &= b+c. \end{aligned}$$

Hence the number of triangles n_I, and the perimeter p_I of I are given by

$$\begin{aligned} n_I &= 2(a+b)(c+d) - a^2 - d^2 \\ p_I &= a + 2(b+c) + d. \end{aligned}$$

Now does a collection of triangles in a regular hexagon, R say of side length

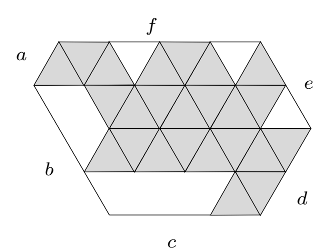


Figure 5: Minimal containing hexagon

m, minimise the perimeter over all possible hexagons with the same area? Now when $a = b = c = d$ we have $n_R = 6m^2$ and $p_R = 6m$ so the question becomes : Does $n_I = n_R$ imply that $p_I \geq p_R$?. That is, does

$$2(a+b)(c+d) - a^2 - d^2 = 6m^2$$

$$\text{imply} \quad a + 2(b+c) + d \geq 6m.$$

First let $\alpha = a+b$, $\beta = c+d$ and $\gamma = e+f$. Then three copies of the above area constraint leads to

$$\begin{aligned} 2\alpha\beta - a^2 - d^2 &= 6m^2 \\ 2\alpha\gamma - b^2 - e^2 &= 6m^2 \\ 2\beta\gamma - c^2 - f^2 &= 6m^2. \end{aligned}$$

Hence

$$2\alpha\beta + 2\alpha\gamma + 2\beta\gamma = 18m^2 + a^2 + b^2 + c^2 + d^2 + e^2 + f^2.$$

Now since $p_I = \alpha + \beta + \gamma$ one obtains

$$p_I^2 = 18m^2 + a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + \alpha^2 + \beta^2 + \gamma^2.$$

Next $p_I = a+b+c+d+e+f$ implies that $a^2+b^2+c^2+d^2+e^2+f^2 \geq 6(\frac{p_I}{6})^2$, and $\alpha^2+\beta^2+\gamma^2 \geq 3(\frac{p_I}{3})^2$ which leads to

$$p_I^2\left(1-\frac{1}{6}-\frac{1}{3}\right) \geq 18m^2$$

$$p_I \geq 6m = p_R.$$

So a complete regular hexagon full of triangles is a minimal configuration i.e. for $n_R = 6m^2$, we have shown that

$$E_{\min}(6m^2) = 9m^2 + 3m.$$

Next we shall show that an incomplete hexagon full of triangles is also a minimal configuration. But first we require the following

Lemma 2 *For a given collection of triangles, C say, let n_C denote the number of triangles and p_C denote the perimeter of C. Then we have*

$$n_C \equiv p_C (\text{mod } 2).$$

Proof : The addition of one triangle to C either increases the perimeter by one or decreases the perimeter by one. So if p_C is even then $p_{C'}$ is odd while if p_C is odd then $p_{C'}$ is even. In other words the addition of one triangle changes the parity of the perimeter. Since an isolated triangle has an odd perimeter the result follows. □

Proposition 2 *: If $n_I = n_{R+t}$ where $1 \leq t \leq 12m+6$ then $p_I \geq p_{R+t}$ where n_{R+t} denotes the number of triangles in a regular hexagon with t triangles added in a spiral manner in the next layer.*

Proof : Consider odd and even t along each of the six sides of the regular hexagon.

Side 1. Now $n_{R+t} = 6m^2 + t$ where $1 \leq t \leq 2m$ while the perimeter is given by

$$p_{R+t} = \begin{cases} 6m+2 & \text{for even } t \\ 6m+1 & \text{for odd } t. \end{cases}$$

For t odd we have $n_{R+t} \geq 6m^2 + 1$ while by symmetry we note that

$$n_I = \frac{1}{3}[2(a+b)(c+d)+2(a+b)(e+f)+2(c+d)(e+f)-a^2-b^2-c^2-d^2-e^2-f^2].$$

If $\alpha = a+b$, $\beta = c+d$, $\gamma = e+f$ and $p_I = \alpha+\beta+\gamma$ then $n_I = n_{R+t}$ implies that

$$2(\alpha\beta+\alpha\gamma+\beta\gamma) \geq 3(6m^2+1)+a^2+b^2+c^2+d^2+e^2+f^2.$$

Therefore we have

$$\begin{aligned} p_I^2 &\geq 18m^2 + 3 + a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + \alpha^2 + \beta^2 + \gamma^2 \\ &\geq 18m^2 + 3 + p_I^2/6 + p_I^2/3 \\ &\geq 36m^2 + 6. \end{aligned}$$

Hence $p_I > 6m$ and so $p_I \geq 6m + 1 = p_R + t$.

For t even we have $n_{R+t} \geq 6m^2 + 2$. Then $n_I = n_{R+t}$ implies that $p_I^2 \geq 6(6m^2 + 2)$ and so as above $p_I > 6m$ thus $p_I \geq 6m + 1$. By the lemma above we must have $p_I \equiv n_I (\text{mod } 2)$ but $n_I \equiv n_{R+t} \equiv 0 (\text{mod } 2)$ hence $p_I \geq 6m + 2 = p_{R+t}$.

Side 2. This time $n_{R+t} = 6m^2 + t$ where $2m + 1 \leq t \leq 4m + 1$ while the perimeter is given by

$$p_{R+t} = \begin{cases} 6m + 2 & \text{for even } t \\ 6m + 3 & \text{for odd } t. \end{cases}$$

For t odd we have $n_{R+t} \geq 6m^2 + 2m + 1$. Then $n_I = n_{R+t}$ implies that

$$\begin{aligned} p_I^2 &\geq 6(6m^2 + 2m + 1) \\ &\geq 36m^2 + 12m + 6 \\ &\geq (6m + 1)^2 + 5. \end{aligned}$$

So as before we find that $p_I \geq 6m + 2$ but $p_I \equiv n_I \equiv 1 (\text{mod } 2)$ implies that $p_I \geq 6m + 3 = p_{R+t}$.

For t even we have $n_{R+t} \geq 6m^2 + 2m + 2$. Then $n_I = n_{R+t}$ implies that

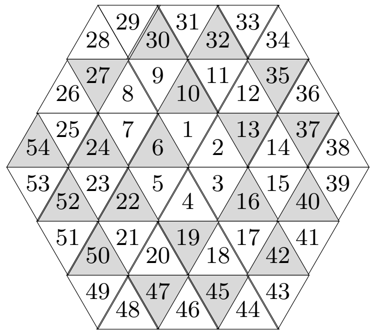
$$\begin{aligned} p_I^2 &\geq 6(6m^2 + 2m + 2) \\ &\geq 36m^2 + 12m + 12 \\ &\geq (6m + 1)^2 + 11. \end{aligned}$$

Hence $p_I \geq 6m + 2 = p_{R+t}$ and the result follows. The other four sides are entirely similar. □

Finally, we can piece together all the above observations to yield:

Proposition 3 *The spiral algorithm applied to the regular triangular tiling provides a minimal configuration for any number of triangles with the number of edges given by*

$$E_{min}(n) = \left\lceil \frac{3n}{2} + \sqrt{\frac{3n}{2}} \right\rceil.$$



First triangle - 3 edges

White triangle - add 2 edges

Shaded triangle - add 1 edge

Figure 6: Triangular spiral algorithm

1.3 Regular Hexagons

Let $H(n,p)$ denote the number of edges contained in a configuration of n unit-hexagons with p edges along the perimeter. Then

$$\begin{aligned} H(n,p) &= \frac{6n-p}{2} + p \\ &= 3n + \frac{p}{2}. \end{aligned}$$

As before comparison with a circle gives us the lower bound

$$H_{\min}(n) \geq 3n + \sqrt{\frac{3\sqrt{3}\pi n}{2}}.$$

One can surround an arbitrary collection of hexagons, C say, with an irregular hexagon, I say, full of unit-hexagons such that the perimeter of I is less than or equal to the perimeter of C. Now if the number of unit-hexagons along each of the sides of I are denoted by a, b, c, d, e, f then the number of equilateral triangles contained in I is given by

$$\begin{aligned} n_I &= 2(\sqrt{3}(a-1) + \sqrt{3}(b-1))(\sqrt{3}(c-1) + \sqrt{3}(d-1)) - (\sqrt{3}(a-1))^2 \\ &\quad - (\sqrt{3}(d-1))^2 + 3(a-1) + 1 + 3(b-1) + 1 + \cdots + 3(f-1) + 1, \\ &= 3(2(a-1+b-1)(c-1+d-1) - (a-1)^2 - (d-1)^2) + 3p - 12. \end{aligned}$$

While the perimeter of I is given by

$$\begin{aligned} p_I &= 2(a-1) + 1 + 2(b-1) + 1 + \cdots + 2(f-1) + 1 \\ &= 2p - 6 \end{aligned}$$

where $p = a + b + c + d + e + f$.

If I is regular with m unit-hexagons along each side then the formulæabove become

$$n_R = 6(3m^2 - 3m + 1), \text{and}$$

$$p_R = 6(2m-1).$$

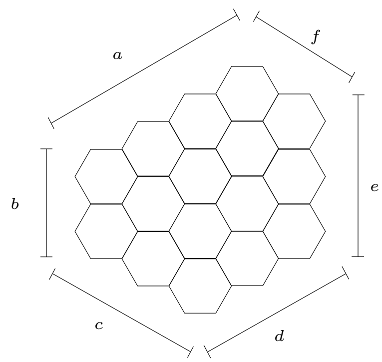


Figure 7: Minimal containing hexagon

Lemma 3 *If $n_I = n_R$ then $p_I \geq p_R$.*

Proof : Let $\alpha = a-1+b-1$, $\beta = c-1+d-1$ and $\gamma = e-1+f-1$ then the above equations become

$$6\alpha\beta = 6(3m^2-3m+1)+3(a-1)^2+3(d-1)^2-(3p-12)$$

$$6\alpha\gamma = 6(3m^2-3m+1)+3(b-1)^2+3(e-1)^2-(3p-12)$$

$$6\beta\gamma = 6(3m^2-3m+1)+3(c-1)^2+3(f-1)^2-(3p-12).$$

Next since $p = \alpha+\beta+\gamma+6$ we get $p_I - 6 = 2(\alpha+\beta+\gamma)$ which leads to

$$\begin{aligned}(p_I-6)^2 =& 4((a-1)^2+(b-1)^2+(c-1)^2+(d-1)^2+(e-1)^2+(f-1)^2) \\ &+ 24(3m^2-3m+1)+4(\alpha^2+\beta^2+\gamma^2)-(12p+48).\end{aligned}$$

But $(a-1)^2+(b-1)^2+(c-1)^2+(d-1)^2+(e-1)^2+(f-1)^2 \geq 6(\frac{p-6}{6})^2$ while $\alpha^2+\beta^2+\gamma^2 \geq 3(\frac{p-6}{3})^2$ so that

$$(p_I-6)^2(1-\frac{1}{3}-\frac{1}{6}) \geq 24(3m^2-3m+1)-6(p_I-6)-24$$

$$(p_I-6)^2+12(p_I-6) \geq 12^2(m^2-m)$$

$$p_I \geq 12m-6 = p_R.$$

□

From this we immediately obtain the minimal number of edges used to construct $3m^2-3m+1$ hexagons, namely:

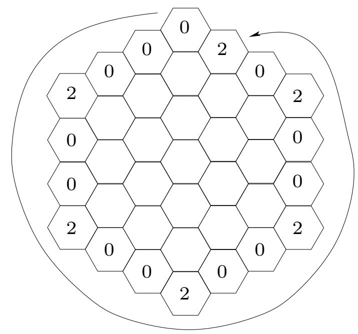


Figure 8: Perimeter change on addition of unit hexagons

Proposition 4 $H_{min}(3m^2 - 3m + 1) = 3(3m^2 - 3m + 1) + 3(2m - 1)$.

As for the square and equilateral tilings if we add unit hexagons to the outside of a complete hexagon the number of edges used remains minimal.

Proposition 5 *Let* $1 \leq t \leq 6m$. *Let* n_{R+t} *be the number of triangles in a regular hexagon with* t *hexagons added around the outside. Let* $n_I = n_{R+t}$. *Then* $p_I \geq p_{R+t}$.

Proof : Consider each of the six sides separately.

Side 1. If $n_{R+t} = 6(3m^2 - 3m + 1) + 6t$ where $1 \leq t \leq m - 1$ then $n_I = n_{R+t}$ implies that $n_I \geq 6(3m^2 - 3m + 2)$. So as before we get

$$\frac{1}{2}(p_I - 6)^2 \geq 24(3m^2 - 3m + 2) - 6(p_I - 6) - 24$$

which leads to the quadratic inequality

$$(p_I - 6)^2 + 12(p_I - 6) - 48(3m^2 - 3m + 1) \geq 0.$$

Hence

$$\begin{aligned} p_I &\geq \sqrt{6^2 + 48(3m^2 - 3m + 1)} \\ &\geq \sqrt{(12m - 6)^2 + 48} \\ &\geq 12m - 5. \end{aligned}$$

But recall that p_I is even which implies that $p_I \geq 12m - 4$. Furthermore $p_{R+t} = 6(2m - 1) + 2$ since only the first hexagon changes the perimeter (by two) while the rest leave it unchanged. So $p_I \geq p_{R+t}$ and the spiral algorithm is minimal down the first side of the hexagon.

Side 2. Now $n_{R+t} = 6(3m^2 - 3m + 1) + 6t$ where $m \leq t \leq 2m - 1$ then $n_I = n_{R+t}$ implies that $n_I \geq 6(3m^2 - 2m + 1)$. So this time we get

$$(p_I - 6)^2 + 12(p_I - 6) - 48(3m^2 - 2m) \geq 0.$$

Hence

$$\begin{aligned} p_I &\geq \sqrt{6^2 + 48(3m^2 - 2m)} \\ &\geq \sqrt{(12m-4)^2 + 20} \\ &\geq 12m - 3. \end{aligned}$$

Again p_I being even implies that $p_I \geq 12m - 2 = p_{R+t}$. The remaining four sides are similar and so the spiral algorithm is minimal for the entire hexagon i.e. for any number of unit hexagons. □

For the regular hexagon we have

$$H_{\min}(n) = \lceil 3n + \sqrt{12n - 3} \rceil.$$

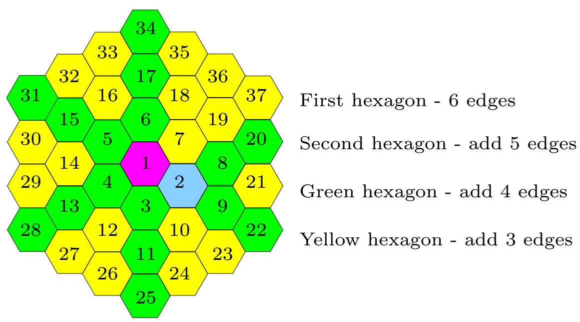


Figure 9: Hexagonal spiral algorithm

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