Volume of an n dimensional simplex

Ralph H. Buchholz and Michael J. Smith

Aug 1996

1 Introduction

In 1992 the first author ([Buc '92]) considered the problem of finding arbitrary tetrahedra with integer edges, faces and volume as well as the generalisation of this problem to n-dimensions. For a 2-dimensional simplex one can simply resort to Heron's formula for the area of a triangle with sides a, b, c given by

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = (a+b+c)/2$ is the semiperimeter. If we square this and expand the right hand side we obtain

$$16Area^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4$$

which can be written in vector-matrix form as

$$16Area^2 = \begin{bmatrix} a^2 & b^2 & c^2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a^2 \\ b^2 \\ c^2 \end{bmatrix}.$$

Now, in 3 dimensions, the volume of an arbitrary triangular pyramid when expressed in matrix form, as $144V^2 = X^t M X$, is not nearly as symmetric as the 2-dimensional case. Furthermore the matrix M no longer contains simply coefficients of appropriate monomials but squares of edge lengths. Compared to this, the 4 dimensional case is a complete mess.

This made the task of extending the results to higher dimensions rather imposing. However, in recent discussions the second author suggested that the symmetry could be restored by using tensor notation. Here we present the results found by exploring that comment.

2 Tensor version for low dimensional cases

The two dimensional case is already symmetric and so simply required translation to tensor notation as

$$16Area^2 = T^{ij} X_i X_j$$

where $X = \begin{bmatrix} a^2 & b^2 & c^2 \end{bmatrix}$ and

$$T^{ij} = \begin{cases} -1 & \text{if } i = j \\ 1 & \text{otherwise.} \end{cases}$$

Throughout this paper we use the Einstein summation convention which requires that there be an implicit summation over the range of any index appearing twice in a term, once as a superscript and once as a subscript.

Next we consider the 3-dimensional case (see Figure 1). The volume of an

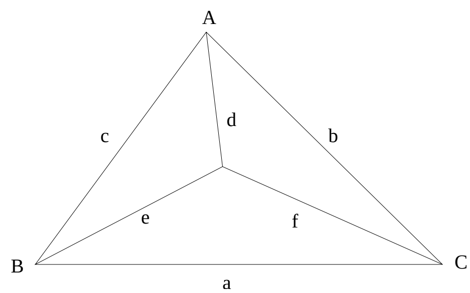

Figure 1: An arbitrary tetrahedron

arbitrary tetrahedron [Som '58] is given by

$$\begin{aligned} 144V^2 =& (a^2 + d^2)(-a^2d^2 + b^2e^2 + c^2f^2) + (b^2 + e^2)(a^2d^2 - b^2e^2 + c^2f^2) \\ &+ (c^2 + f^2)(a^2d^2 + b^2e^2 - c^2f^2) - a^2b^2c^2 - a^2e^2f^2 - b^2d^2f^2 - c^2d^2e^2. \end{aligned}$$

After a little experimentation (which involved recognising the monomials, of the expression above, as graphs related to a labelled K_4) we managed to convert this case into a symmetric form, namely,

$$V = \frac{1}{12\sqrt{2}} \left(T^{ijk} X_i X_j X_k\right)^{1/2}$$

where the tensor is a $6 \times 6 \times 6$ coefficient array of the monomials given by

$$
T = \begin{array}{c}
\begin{bmatrix} 0 & 0 & 0 & -2/3 & 0 & 0 \\ 0 & 0 & -1/3 & 1/3 & 1/3 & 0 \\ 0 & -1/3 & 0 & 1/3 & 0 & 1/3 \\ -2/3 & 1/3 & 1/3 & -2/3 & 1/3 & 1/3 \\ 0 & 1/3 & 0 & 1/3 & 0 & -1/3 \\ 0 & 0 & 1/3 & 1/3 & -1/3 & 0 \end{bmatrix}
\begin{bmatrix} 0 & 0 & -1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & -2/3 & 0 \\ -1/3 & 0 & 0 & 0 & 1/3 & 1/3 \\ 1/3 & 0 & 0 & 0 & 1/3 & -1/3 \\ 1/3 & -2/3 & 1/3 & 1/3 & -2/3 & 1/3 \\ 0 & 0 & 1/3 & -1/3 & 1/3 & 0 \end{bmatrix} \\
\begin{bmatrix} 0 & -1/3 & 0 & 1/3 & 0 & 1/3 \\ -1/3 & 0 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & -2/3 \\ 1/3 & 0 & 0 & 0 & -1/3 & 1/3 \\ 0 & 1/3 & 0 & -1/3 & 0 & 1/3 \\ 1/3 & 1/3 & -2/3 & 1/3 & 1/3 & -2/3 \end{bmatrix}
\begin{bmatrix} -2/3 & 1/3 & 1/3 & -2/3 & 1/3 & 1/3 \\ 1/3 & 0 & 0 & 0 & 1/3 & -1/3 \\ 1/3 & 0 & 0 & 0 & -1/3 & 1/3 \\ -2/3 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & -1/3 & 0 & 0 & 0 \\ 1/3 & -1/3 & 1/3 & 0 & 0 & 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 1/3 & 0 & 1/3 & 0 & -1/3 \\ 1/3 & -2/3 & 1/3 & 1/3 & -2/3 & 1/3 \\ 0 & 1/3 & 0 & -1/3 & 0 & 1/3 \\ 1/3 & 1/3 & -1/3 & 0 & 0 & 0 \\ 0 & -2/3 & 0 & 0 & 0 & 0 \\ -1/3 & 1/3 & 1/3 & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} 0 & 0 & 1/3 & 1/3 & -1/3 & 0 \\ 0 & 0 & 1/3 & -1/3 & 1/3 & 0 \\ 1/3 & 1/3 & -2/3 & 1/3 & 1/3 & -2/3 \\ 1/3 & -1/3 & 1/3 & 0 & 0 & 0 \\ -1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & -2/3 & 0 & 0 & 0 \end{bmatrix}
\end{array}
$$

The observant reader will have noticed that the sum of all the entries of the tensor is 4=3+1. This is intimately connected to the degenerate case of regular simplices which we consider later. Also, since this tensor is symmetric about the planes $i = j$, $i = k$ and $j = k$ we have considerable redundancy and so can compress the description of it substantially. In fact we found that

$$
T^{(\Gamma)} = \begin{cases} -2/3 & \Gamma \cong K_2 + P_1 \\ -1/3 & \Gamma \cong K_3 \\ 1/3 & \Gamma \cong P_3 \\ 0 & \text{otherwise} \end{cases}
$$

where Γ is a graph containing 3 edges (possibly repeated) chosen from a labelled complete graph on 4 vertices. The only graphs providing a non-zero contribution are the three cases shown, namely a 2-circuit with an edge, a 3-circuit, and a 3-edge path. Notice that none of these have a vertex with valency larger than 2. We conjectured that this would be true in general.

3 General case

First we make a number of notational definitions. Recall that for an arbitrary graph G the set of edges and the set of vertices are denoted by EG and VG respectively.

Definition Take formal sums (allowing repetition) of edges of some graph G to be

$$\Gamma = \sum_{e \in EG} c_e \cdot e$$

where $c_e \in \mathbb{N}$.

Of course, we are secretly thinking of these formal sums as graphs in their own right. However, they are not necessarily subgraphs of G due to the fact that multiple copies of an edge of G are allowed in Γ.

Definition Let $\mathcal{L}_m(G)$ denote the set of all such m-long formal sums of edges of G, i.e.

$$\mathcal{L}_m(G) = \left\{ \sum_{e \in EG} c_e \cdot e \mid \sum c_e = m \right\}.$$

With these definitions out of the way we can state the general result.

Theorem 1 *The volume of an n-dimensional simplex is given by*

$$\mathcal{V}(n) = \frac{1}{n! 2^{n/2}} \left(T^{i_1 \cdots i_n} X_{i_1} \cdots X_{i_n} \right)^{1/2}$$

where $X = \left[a_1^2 \cdots a_{n(n+1)/2}^2 \right]$ is a vector containing the squares of the edge lengths, and for all $\Gamma \in \mathcal{L}_n(K_{n+1})$ the tensor entry corresponding to Γ is given by

$$T^\Gamma = \begin{cases} -1^{\#loops(\Gamma)} \times 2^{\#components(\Gamma)}/n! & \text{if valency}(\Gamma) \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

The maximal valency constraint implies that each Γ is just a disjoint union of paths and circuits. Before proceeding with the proof of this theorem we first show it in action in the 6-dimensional case. For a regular n-simplex (i.e. all

Γ	# labellings	# orderings	regular contribution
C_6	420	720	-840
$C_5 \cup P_1$	252	720	-1008
$C_4 \cup P_2$	315	720	-1260
$C_4 \cup C_2$	315	360	630
$C_3 \cup C_3$	70	720	280
$C_3 \cup P_3$	420	720	-1680
$C_3 \cup C_2 \cup P_1$	210	360	840
P_6	2520	720	5040
$P_4 \cup C_2$	1260	360	-2520
$P_2 \cup C_2 \cup C_2$	315	180	630
$C_2 \cup C_2 \cup C_2$	105	90	-105

Table 1: Valency 2 graphs for a 6-dimensional simplex

edges equal in length) it is relatively easy to prove ([Buc '92, p.367]) that the volume is given by

$$\mathcal{V}(n) = \frac{a^n}{n!}\sqrt{\frac{n+1}{2^n}}$$

where a is the common edge length. Now the regular contribution is obtained by #labellings $\times$ #orderings $\times$ T^Γ and so for $n = 6$ we must (and in fact do) have

$$\sum \text{regular contribution} = 7.$$

Using a little graph theory we can in fact directly prove the following.

Theorem 2 *For the tensor defined in the previous theorem we have*

$$\sum T^{i_1 \cdots i_n} = n+1$$

where the sum is taken over all the tensor entries.

Proof : Let $l(\Gamma)$, $l_2(\Gamma)$, $c(\Gamma)$, $r(\Gamma)$ and $s(\Gamma)$ denote the # circuits, # 2-circuits, # components, rank and co-rank of Γ respectively. Then

$$\sum T^{i_1 \cdots i_n} = \sum_{\Gamma \in \mathcal{L}_n(K_{n+1})} T^\Gamma \frac{n!}{2^{l_2(\Gamma)}}.$$

Now replacing T^Γ on the right leads to

$$\begin{aligned}
&\sum T^{i_1 \cdots i_n} \\
&= \sum_{\Gamma \in \mathcal{L}_n(K_{n+1})} (-1)^{l(\Gamma)} 2^{c(\Gamma) - l_2(\Gamma)} \\
&= (-1)^n \sum_{\Gamma \in \mathcal{L}_n(K_{n+1})} (-1)^{r(\Gamma)} 2^{s(\Gamma) + |V\Gamma| - |E\Gamma| - l_2(\Gamma)} \\
&= \frac{(-1)^n}{2^n} \sum_{\Gamma \in \mathcal{L}_n(K_{n+1})} \frac{(-1)^{r(\Gamma)} 2^{s(\Gamma)+|V\Gamma|}}{2^{l_2(\Gamma)}} \\
&= \frac{(-1)^n}{2^n} \left\{ \sum_{\substack{\Gamma \in \mathcal{L}_n(K_{n+1}) \\ |V\Gamma| = n+1}} \frac{(-1)^{r(\Gamma)} 2^{s(\Gamma)+|V\Gamma|}}{2^{l_2(\Gamma)}} + \sum_{\substack{\Gamma \in \mathcal{L}_n(K_{n+1}) \\ |V\Gamma| = n}} \frac{(-1)^{r(\Gamma)} 2^{s(\Gamma)+|V\Gamma|}}{2^{l_2(\Gamma)}} \right\} \\
&= (-1)^n \left\{ 2 \sum_{\substack{\Gamma \in \mathcal{L}_n(K_{n+1}) \\ |V\Gamma| = n+1}} \frac{(-1)^{r(\Gamma)} 2^{s(\Gamma)}}{2^{l_2(\Gamma)}} + \sum_{\substack{\Gamma \in \mathcal{L}_n(K_{n+1}) \\ |V\Gamma| = n}} \frac{(-1)^{r(\Gamma)} 2^{s(\Gamma)}}{2^{l_2(\Gamma)}} \right\} \\
&= (-1)^n \left\{ 2(-1)^n \frac{n(n+1)}{2} + (-1)^{n-1}(n^2 - 1) \right\} \\
&= n+1.
\end{aligned}$$

•proof of general case.

4 References

1. Buchholz, Ralph H. *Perfect Pyramids* Bulletin Australian Mathematical Society, vol. 45, no. 3, 1992.

2. Sommerville, D. M. Y., *An Introduction to the Geometry of n Dimensions* Dover, New York, 1958.