On a geometric proposition*

L. Fejes Tóth

1940

In the following we prove the proposition:

Postulate 1 *Let $\mathfrak{S}_2, \mathfrak{S}_3, \ldots, \mathfrak{S}_n, \ldots$ be a sequence of configurations of points, composed of $2, 3, \ldots, n, \ldots$ points respectively, and which all lie in a region of area T. Then in every $\mathfrak{S}_n$ one can choose a pair of points with a gap of d_n in such a way that*

$$\limsup_{n\to\infty} n d_n^2 \leq \frac{2\sqrt{3}}{3} T$$

ensues.

The constant $\frac{2\sqrt{3}}{3}$ cannot be replaced by anything smaller.[1]

To prove our proposition we find the pair of points in $\mathfrak{S}_n$ with the smallest gap, d_n, and place circular disks $\mathfrak{K}_i$ $(i = 1, 2, \ldots, n)$ with a common radius $\rho_n = \frac{\sqrt{3}}{3} d_n$ at each point in the configuration with the point at the centre of the disk.[2] It is clear that no point in the plane can be covered by more than two circular disks. The total area of $T_n = \sum_{i=1}^{n} \mathfrak{K}_i$, i.e. that of the circular disk covered region of the plane, is therefore

$$T_n = \sum_{i=1}^{n} (t_i^1 + \frac{1}{2} t_i^2)$$

*Translation of "Über einen geometrischen Satz" by Ralph H. Buchholz : April 8, 2006

[1] One considers a pair of points with the smallest separation amongst a configuration with a given number of points, in a circular or square region. Mr D. Lázár made me aware of the problem of determining the configuration of points for which this gap attains a maximum value.

[2] If one considers an equilateral triangle with side-length d_n then $\rho_n = \frac{\sqrt{3}}{3} d_n$ is the radius of the circumcircle.

where t_i^1 denotes the singly covered area and t_i^2 denotes the doubly covered piece of the area of $\mathfrak{K}_i$.[3] However, for every i it turns out that $t_i^1 + \frac{1}{2}t_i^2 \geq \frac{3\sqrt{3}}{2}\rho_n^2$. This follows from the fact that t^2 attains its maximum when six other circles intrude into $\mathfrak{K}_i$. The centres of these circles lie at the vertices of a regular hexagon which is concentric with $\mathfrak{K}_i$ and has side length d_n (see Figure 1).[4] We therefore obtain $T_n \geq \frac{3\sqrt{3}}{2}n\rho_n^2$, which since $T \geq \limsup_{n\to\infty} T_n$

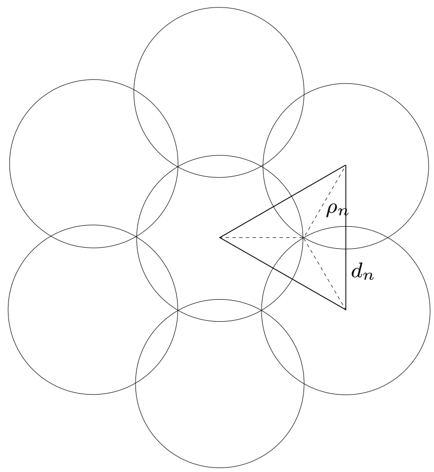

Figure 1:

[3]Let $\mathfrak{G}_1, \mathfrak{G}_2, \ldots, \mathfrak{G}_k$ denote a finite number of arbitrary partial or fully overlapping regions, then the content of the combined area is

$$T = \sum_{i=1}^{k} \left(t_i^1 + \frac{1}{2}t_i^2 + \ldots + \frac{1}{k}t_i^k \right),$$

where t_i^l denotes the l-fold covered region of $\mathfrak{G}_i$.

[4]$\mathfrak{K}_i$ can also possess 7 overlapping circular disks. However, these can only cover an extremely small piece of $\mathfrak{K}_i$. To clarify this, we denote the centres of $\mathfrak{K}_i$ and the 7 overlapping circular disks respectively by $O, O_1, O_2, \ldots, O_7$. It turns out that $d_n = \sqrt{3}\rho_n \leq \overline{OO_l} \leq 2\rho_n$, $\overline{O_lO_{l+1}} \geq \sqrt{3}\rho_n$ $(l = 1, 2, \ldots, 7; O_8 \equiv O_1)$. From this, it directly follows that the septagon $O_1O_2\ldots O_7$ is convex. Thus for an arbitrary l the interval $\overline{OO_l}$ attains its minimum if $O_1O_2\ldots O_7$ is equilateral and for $k \neq l$ we have $\overline{OO_k} = 2\rho_n$. Therefore one obtains, through a simple calculation, the inequality

$$\overline{OO_l} \geq [(4\cos^2 5\alpha - 1)^{1/2} - 2\cos 5\alpha]\rho_n > 1.97\rho_n$$

where $\sin\alpha = \frac{\sqrt{3}}{4}$; from which one easily obtains the completely rough estimate $t_i^2 < \frac{1}{40}\rho_n^2$. Furthermore, it is clear that $\mathfrak{K}_i$ can be overlapped by at most 7 circles, since in an annulus bounded by the concentric circles of radii $\sqrt{3}\rho_n$ and $2\rho_n$ one can place at most 7 points such that the smallest gap between any two is $\sqrt{3}\rho_n$.

leads to the inequality $T \geq \frac{\sqrt{3}}{2} \limsup_{n\to\infty} nd_n^2$, as above.

Therefore, if we wish to distribute n points in a region in such a way that each point lies as far as possible from every other point then one must "tile" the region with congruent equilateral triangles with side lengths $d_n \sim \sqrt{\frac{2\sqrt{3}T}{3n}}$ and place the points at the vertices of the triangles.

It is very likely that the shortest path connecting all the points attains its maximum length by using the same asymptotic distribution. This leads to the following conjecture (which has the previous statement as a corollary): If n points lie in a region of area T then they can be connected with a path of length $L_n \sim \sqrt{\frac{2\sqrt{3}T}{3n}}$.

We do not wish to consider the analogous problems in 3-space too closely. However, consider the point distribution from above, i.e. the triangular tiling, and construct a regular tetrahedron above every triangle by a lift and parallel translation of the distribution. If we take the vertices of the tetrahedra which lie in the original plane then we recover the original point distribution. If one continues in such a way then one expects to obtain a distribution of points in 3-space which asymptotically represents the best possible distribution for the corresponding problem in three dimensions.

(Received 14 August 1939.)